Sketch-as-proof*

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Abstract. This paper presents an extension of Gentzen's **LK**, called $\mathbf{L_{PG}K}$, which is suitable for expressing projective geometry and for deducing theorems of plane projective geometry. The properties of this calculus are investigated and the cut elimination theorem for $\mathbf{L_{PG}K}$ is proven. A formulization of sketches is presented and the equivalence between sketches and formal proofs is demonstrated.

1 Introduction

Sketches are very useful things to illustrate the facts of a proof and to make the idea of a proof transparent. But sketches need not only be just a hint, they can, in certain cases, be regarded as a proof by itself. Projective geometry[1] is best for analyzing this relation between sketches and proofs.

The purpose of this paper is to bring an idea of what a sketch can do and to explain the relations between sketches and proofs. Therefore we extend Gentzen's **LK** to $\mathbf{L_{PG}K}$ and use the properties of $\mathbf{L_{PG}K}$ to formalize the concept of a sketch. We will then present a result on the equivalence of sketches and proofs.

2 A Short Introduction to Projective Geometry

The root of projective geometry is the parallel postulate introduced by Euclid (*c.* 300 B.C.). The believe in the absolute truth of this postulate remains unshakable till the 19th century when the founders of non-Euclidean geometry—Carl Friedrich Gauss (1777–1855), Nicolai Ivanovitch Lobachevsky (1793–1856), and Johann Bolyai (1802–1860)—concluded independently that a consistent geometry denying Euclid's parallel postulate could be set up. Nevertheless projective geometry was developed as an extension of Euclidean geometry; i.e., the parallel postulate was still used and a line was added to the Euclidean plane to contain the "ideal points", which are the intersection of parallel lines. Not till the end of the 19th century and the beginning of the 20th century, through the work of Felix Klein (1849–1925), Oswald Veblen (1880–1960), David Hilbert, and others, projective geometry was seen to be independent of the theory of parallels.

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[1] We will understand under "projective geometry" the *plane* projective geometry and will loose the "plane" for simplicity.

Projective geometry was then developed as an abstract science based on its own set of axioms. For a more extensive discussion see [4], [3].

The projective geometry deals, like the Euclidian geometry, with points and lines. These two elements are primitives, which aren't further defined. Only the axioms tell us about their properties. We will use the expression *Point* (note the capital P) for the objects of projective geometry and *points* as usual for e.g. a point in a plane. The same applies to *Line* and *line*. The only predicate beside the equality is called *Incidence* and puts up a relation between Points and Lines.

Furthermore we must give some axioms to express certain properties of Points and Lines and to specify the behavior of the incidence on Points and Lines:

(PG1) For every two distinct Points there is one and only one Line, so that these two Points incide with this Line.
(PG2) For every two distinct Lines there is one and only one Point, so that this Point incides with the two Lines[2].
(PG3) There are four Points, which never incide with a Line defined by any of the three other Points.

2.1 Examples for Projective Planes

The projective closed Euclidean plane Π_{EP} The easiest approach to projective geometry is via the Euclidean plane. If we add one Point "at infinity" to each line and one "ideal Line", consisting of all these "ideal Points", it follows that two Points determine exactly one Line and two distinct Lines determine exactly one Point[3]. So the axioms are satisfied.

This projective plane is called Π_{EP} and has a lot of other interesting properties, especially that it is a classical projective plane.

The minimal Projective Plane One of the basic properties of projective planes is the fact, that there are seven distinct Points. Four Points satisfying axiom (PG3) and the three diagonal Points $([A_0B_0][C_0D_0]) =: D_1$ etc. (see. fig. 1). If we can set up a relation of incidence on these Points such as that the axioms (PG1) and (PG2) are satisfied, then we have a minimal projective plane. Fig. 1 defines such an incidence-table. This table has to be read carefully: The straight lines and the circle symbolize the Lines and the labeled points the Points of the minimal projective plane. There are no more Points, Lines, especially no intersections as the holes in the figure should suggest.

3 The Calculus $\mathbf{L_{PG}K}$

The calculus $\mathbf{L_{PG}K}$ is based on Gentzen's **LK**, but extends it by certain means. The usual notations as found in [6] are used.

[2] "one and only one" can be replaced by "one", because the fact that there is not more than one Point can be proven from axiom (PG1).

[3] More precise: The "ideal Points" are the congruence classes of the lines with respect to the parallel relation and the "ideal Line" is the class of these congruence classes.

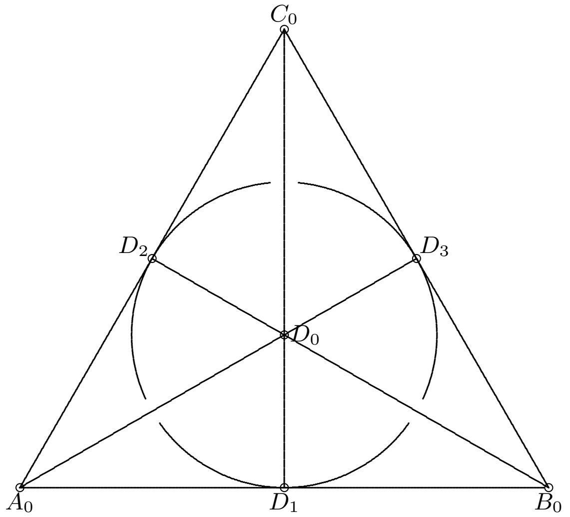

Fig. 1. Incidence Table for the minimal Projective Plane

3.1 The Language $\mathbf{L_{PG}}$ for $\mathbf{L_{PG}K}$

The language for $\mathbf{L_{PG}K}$ is a type language with two types, Points and Lines. These two types will be denoted with $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{L}}$, respectively. There are four individual constants of type $\tau_{\mathcal{P}}$: A_0, B_0, C_0, D_0, two function constants (the type is given in parenthesis): con:$[\tau_{\mathcal{P}}, \tau_{\mathcal{P}} \rightarrow \tau_{\mathcal{L}}]$, intsec:$[\tau_{\mathcal{L}}, \tau_{\mathcal{L}} \rightarrow \tau_{\mathcal{P}}]$, and two predicate constants (the type is given in parenthesis): $\mathcal{I}$:$[\tau_{\mathcal{P}}, \tau_{\mathcal{L}}]$, $=$.

There are free and bound variables of type $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{L}}$, which are P_i (free Points), X_i (bound Points), g_i (free Lines), x_i (bound Lines) and we will use the logical symbols $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\supset$ (implies), $\forall_{\tau_{\mathcal{P}}}$ (for all Points), $\forall_{\tau_{\mathcal{L}}}$ (for all Lines), $\exists_{\tau_{\mathcal{P}}}$ (there exists a Point), $\exists_{\tau_{\mathcal{L}}}$ (there exists a Line).

The constants $A_0, \ldots, D_0$ are used to denote the four Points obeying (PG3). We will use further capital letters, with or without sub– and superscripts, for Points[4] and lowercase letters, with or without sub– and superscripts, for Lines.

Furthermore we will use the notation $[PQ]$ for the connection $\mathrm{con}(P, Q)$ of two Points and the notation (gh) for the intersection $\mathrm{intsec}(g, h)$ of two Lines to agree with the classical notation in projective geometry. Finally $\mathcal{I}(P, g)$ will be written $P\mathcal{I}g$.

We also lose the subscript $_{\tau_{\mathcal{P}}}$ and $_{\tau_{\mathcal{L}}}$ in $\forall_{\tau_{\mathcal{P}}}, \ldots$, since the right quantifier is easy to deduce from the bound variable.

The formulization of terms, atomic formulas and formulas is a standard technique and can be found in [6].

[4] Capital letters are also used for formulas, but this shouldn't confuse the reader, since the context in each case is totally different.

3.2 The Rules and Initial Sequents of $\mathbf{L_{PG}K}$

Definition 1. *A* logical initial sequent *is a sequent of the form* $A \rightarrow A$*, where* A *is atomic.*
The mathematical initial sequents *are formulas of one of the following forms:*

1. $\rightarrow P\mathcal{I}[PQ]$ *and* $\rightarrow Q\mathcal{I}[PQ]$.
2. $\rightarrow (gh)\mathcal{I}g$ *and* $\rightarrow (gh)\mathcal{I}h$.
3. $\rightarrow x = x$ *where* x *is a free variable.*

The initial sequents for $\mathbf{L_{PG}K}$ *are the logical initial sequents and the mathematical initial sequents.*

The first two clauses are nothing else then (PG1) and (PG2). (PG3) is realized by a rule. The fact that $X = Y \rightarrow$ for $X, Y \in \{A_0, B_0, C_0, D_0\}$ and $X \neq Y$ can be deduced from this rule. The *rules for* $\mathbf{L_{PG}K}$ are structural rules, logical rules, cut rule (taken from **LK** for many-sorted languages), the following equality rules:

$$\frac{\Gamma \rightarrow \Delta, s = t \quad s = u, \Gamma \rightarrow \Delta}{t = u, \Gamma \rightarrow \Delta} \text{ (trans:left)}$$

$$\frac{\Gamma \rightarrow \Delta, s = t \quad \Gamma \rightarrow \Delta, s = u}{\Gamma \rightarrow \Delta, t = u} \text{ (trans:right)}$$

$$\frac{s = t, \Gamma \rightarrow \Delta}{t = s, \Gamma \rightarrow \Delta} \text{ (symm:left)} \qquad \frac{\Gamma \rightarrow \Delta, s = t}{\Gamma \rightarrow \Delta, t = s} \text{ (symm:right)}$$

$$\frac{\Gamma \rightarrow \Delta, s = t \quad s\mathcal{I}u, \Gamma \rightarrow \Delta}{t\mathcal{I}u, \Gamma \rightarrow \Delta} \text{ (id-}\mathcal{I}_{\tau_\mathcal{P}}\text{:left)}$$

$$\frac{\Gamma \rightarrow \Delta, s = t \quad \Gamma \rightarrow \Delta, s\mathcal{I}u}{\Gamma \rightarrow \Delta, t\mathcal{I}u} \text{ (id-}\mathcal{I}_{\tau_\mathcal{P}}\text{:right)}$$

$$\frac{\Gamma \rightarrow \Delta, u = v \quad s\mathcal{I}u, \Gamma \rightarrow \Delta}{s\mathcal{I}v, \Gamma \rightarrow \Delta} \text{ (id-}\mathcal{I}_{\tau_\mathcal{L}}\text{:left)}$$

$$\frac{\Gamma \rightarrow \Delta, u = v \quad \Gamma \rightarrow \Delta, s\mathcal{I}u}{\Gamma \rightarrow \Delta, s\mathcal{I}v} \text{ (id-}\mathcal{I}_{\tau_\mathcal{L}}\text{:left)}$$

$$\frac{\Gamma \rightarrow \Delta, s = t}{\Gamma \rightarrow \Delta, [su] = [tu]} \text{ (id-con:1)} \qquad \frac{\Gamma \rightarrow \Delta, u = v}{\Gamma \rightarrow \Delta, [su] = [sv]} \text{ (id-con:2)}$$

$$\frac{\Gamma \rightarrow \Delta, g = h}{\Gamma \rightarrow \Delta, (tg) = (th)} \text{ (id-int:1)} \qquad \frac{\Gamma \rightarrow \Delta, g = h}{\Gamma \rightarrow \Delta, (gt) = (ht)} \text{ (id-int:2)}$$

and the mathematical rules: (PG1-ID) and (Erase)

$$\frac{\Gamma \rightarrow \Delta, P\mathcal{I}g \quad \Gamma \rightarrow \Delta, Q\mathcal{I}g \quad P = Q, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, [PQ] = g} \text{ (PG1-ID)}$$

$$\frac{\Gamma \rightarrow \Delta, X\mathcal{I}[YZ]}{\Gamma \rightarrow \Delta} \text{ (Erase)}$$

where $\neq(X, Y, Z)$ and $X, Y, Z \in \{A_0, B_0, C_0, D_0\}$

Finally proofs are defined as usual.

3.3 Sample Proofs in $\mathbf{L_{PG}K}$

The Diagonal-points We will proof that the diagonal-point D_1 (cf. 2.1) is distinct from $A_0, \ldots, D_0$.

$$\dfrac{\begin{matrix}\vdots \\ \to A_0 \neq D_1\end{matrix} \quad \begin{matrix}\vdots \\ \to B_0 \neq D_1\end{matrix} \quad \begin{matrix}\vdots \\ \to C_0 \neq D_1\end{matrix} \quad \begin{matrix}\vdots \\ \to D_0 \neq D_1\end{matrix}}{\to \neq (A_0, B_0, C_0, D_0, D_1)}$$

Each of the proof-parts is similar to the following for $\to A_0 \neq D_1$

$$\dfrac{\dfrac{\dfrac{A_0 = ([A_0B_0][C_0D_0]) \to A_0 = ([A_0B_0][C_0D_0]) \quad \to ([A_0B_0][C_0D_0])\mathcal{I}[C_0D_0]}{A_0 = ([A_0B_0][C_0D_0]) \to A_0\mathcal{I}[C_0D_0]}\ \text{(atom)}}{A_0 = ([A_0B_0][C_0D_0]) \to}\ \text{(Erase)}}{\to A_0 \neq ([A_0B_0][C_0D_0])}\ (\neg\text{:right})$$

Identity of the Intersection-point We will proof the fact, that there is only one intersection-point of g and h, i.e, the dual fact of (PG1-ID).

$$\dfrac{\dfrac{P\mathcal{I}g \to P\mathcal{I}g \quad \to (gh)\mathcal{I}g \quad P = (gh) \to P = (gh)}{P\mathcal{I}g \to P = (gh), [P(gh)] = g}\ \text{(atom)} \qquad \dfrac{P\mathcal{I}h \to P\mathcal{I}h \quad \to (gh)\mathcal{I}h \quad P = (gh) \to P = (gh)}{P\mathcal{I}h \to P = (gh), [P(gh)] = h}\ \text{(atom)}}{P\mathcal{I}g, P\mathcal{I}h \to P = (gh), g = h}\ \text{(atom)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{P\mathcal{I}g, P\mathcal{I}h \to P = (gh), g = h}{g \neq h, P\mathcal{I}g, P\mathcal{I}h \to P = (gh)}\ (\neg\text{:left})}{P\mathcal{I}g \wedge P\mathcal{I}h \wedge g \neq h \to P = (gh)}\ (\wedge\text{:left})}{\to P\mathcal{I}g \wedge P\mathcal{I}h \wedge g \neq h \supset P = (gh)}\ (\supset\text{:right})}{\to (\forall X)(\forall u)(\forall v)(X\mathcal{I}u \wedge X\mathcal{I}v \wedge u \neq v \supset X = (uv))}\ (\forall\text{:right})$$

4 On the Structure of Proofs in $\mathbf{L_{PG}K}$

4.1 The Cut Elimination Theorem for $\mathbf{L_{PG}K}$

We will refer to the equality rules, (PG1-ID) and (Erase) as (atom)-rules, because they only operate on atomic formulas and therefore they can be shifted above any logical rule (see Step 1 below). We will now transform any given proof in $\mathbf{L_{PG}K}$ step by step into another satisfying some special conditions, especially that the new one contains no (Cut).

First we reduce the problem to a special class of proofs, the proofs in normal form[5]. A proof in this class is split into two parts $\mathcal{P}_1$ and $\mathcal{P}_2$

$$\begin{matrix} \vdots\ \mathcal{P}_1 \\ \vdots\ \mathcal{P}_2 \\ \Pi \to \Gamma \end{matrix}$$

[5] This nomenclature is used only in this context and has no connection with any other "normalization".

where $\mathcal{P}_1$ is an (atom)-part with (atom)- and structural rules only and $\mathcal{P}_2$ is a logical part with logical and structural rules only. In the first part geometry is practiced in the sense that in this part the knowledge about projective planes is used. The second part is a logical part connecting the statements from the geometric part to more complex statements with logical connectives. It is easy to see, that for every proof in $\mathbf{L_{PG}K}$ there is a proof in normal form of the same endsequent.

Lemma 1. *For every proof in normal form with only one cut there is a proof in normal form of the same endsequent without a cut.*

PROOF (Sketch, detailed exposition in [1]):

STEP 1: We will start with the cut-elimination procedure as usual in **LK** as described e.g. in [6]. This procedure shifts a cut higher and higher till the cut is at an axiom where it can be eliminated trivially. Since in our case above all the logical rules there is the (atom)-part, the given procedure will only shift the cut in front of this part.

STEP 2: Now the cut is already in front of the (atom)-part:

$$\dfrac{\begin{matrix} \vdots\, \mathcal{P}_1 \\ \Pi_1 \to \Gamma_1, P(t,u) \end{matrix} \quad \begin{matrix} \vdots\, \mathcal{P}_2 \\ P(t,u), \Pi_2 \to \Gamma_2 \end{matrix}}{\Pi \to \Gamma}\ (\text{Cut})$$

First all the inferences not operating on the cut-formulas or one of its predecessors are shifted under the cut-rule. Then the rule from the right branch over the cut-rule are shifted on the left side by applying the dual rules[6] in inverse order. Finally we get on the right side either a logical axiom or a mathematical axiom. The case of a logical axiom is trivial, in case of the mathematical axiom the rules from the left side are applied in inverse order on the antecedent of the mathematical axiom which yields a cut-free proof. □

EXAMPLE: A trivial example should explain this method: The proof

$$\dfrac{\dfrac{x_2 = x_3 \to x_2 = x_3 \qquad \dfrac{x_1 = x_2 \to x_1 = x_2 \quad x_1 = u \to x_1 = u}{x_1 = x_2, x_1 = u \to x_2 = u}}{x_2 = x_3, x_1 = x_2, x_1 = u \to x_3 = u} \qquad x_3 = u \to}{x_2 = x_3, x_1 = x_2, x_1 = u \to}\ (\text{Cut})$$

will be transformed to

$$\dfrac{x_1 = x_2 \to x_1 = x_2 \qquad \dfrac{x_2 = x_3 \to x_2 = x_3 \quad x_3 = u \to}{x_2 = x_3, x_2 = u \to}}{x_1 = x_2, x_2 = x_3, x_1 = u \to}$$

♡

[6] E.g. (trans:left) and (trans:right) are dual rules

Theorem 1 (Cut Elimination for $\mathbf{L_{PG}K}$). *If there is a proof of a sequent $\Pi \to \Gamma$ in $\mathbf{L_{PG}K}$, then there is also a proof without a cut.*

Proof: By the fact that everything above a given sequent is a proof of this sequent and by using Lemma 1 and induction on the number of cuts in a proof we could eliminate one cut after another and end up with a cut-free proof. □

Example: We will now present an example proof and the corresponding proof without a cut. We want to prove that for every line there is a point not on that line, in formula: $(\forall g)(\exists X)(X \not\mathcal{I} g)$.

We will first give the proof in words and then in $\mathbf{L_{PG}K}$.

Proof: (Words) When $A_0 \not\mathcal{I} g$ then take A_0 for X. Otherwise $A_0 \mathcal{I} g$. Next if $B_0 \not\mathcal{I} g$ take B_0 for X. If also $B_0 \mathcal{I} g$ then take C_0, since when A_0 and B_0 lie on g, then $g = [A_0 B_0]$ and $C_0 \not\mathcal{I} [A_0 B_0] = g$ by (PG3). □

Proof: ($\mathbf{L_{PG}K}$)

$$\cfrac{\cfrac{\cfrac{\cfrac{A_0\mathcal{I}g \to A_0\mathcal{I}g}{\to A_0\mathcal{I}g, A_0\not\mathcal{I}g}}{\to A_0\mathcal{I}g \vee A_0\not\mathcal{I}g} \qquad \cfrac{\cfrac{\cfrac{A_0\mathcal{I}g \to A_0\mathcal{I}g}{A_0\not\mathcal{I}g \to A_0\not\mathcal{I}g}}{A_0\not\mathcal{I}g \to (\exists X)(X\not\mathcal{I}g)} \qquad \cfrac{\cfrac{\cfrac{B_0\mathcal{I}g \to B_0\mathcal{I}g}{\to B_0\mathcal{I}g, B_0\not\mathcal{I}g}}{\to B_0\mathcal{I}g \vee B_0\not\mathcal{I}g} \qquad \cfrac{\cfrac{\cfrac{B_0\mathcal{I}g \to B_0\mathcal{I}g}{B_0\not\mathcal{I}g \to B_0\not\mathcal{I}g}}{B_0\not\mathcal{I}g \to (\exists X)(X\not\mathcal{I}g)} \qquad \begin{matrix}\vdots\ \Pi_1 \\ A_0\mathcal{I}g, B_0\mathcal{I}g \to (\exists X)(X\not\mathcal{I}g)\end{matrix}}{B_0\mathcal{I}g \vee B_0\not\mathcal{I}g, A_0\mathcal{I}g \to (\exists X)(X\not\mathcal{I}g)}}{A_0\mathcal{I}g \to (\exists X)(X\not\mathcal{I}g)}\ (\text{Cut})}{A_0\mathcal{I}g \vee A_0\not\mathcal{I}g \to (\exists X)(X\not\mathcal{I}g)}}{\to (\exists X)(X\not\mathcal{I}g)}\ (\text{Cut})}{\to (\forall g)(\exists X)(X\not\mathcal{I}g)}$$

Π_1 :

$$\cfrac{\cfrac{\cfrac{A_0\mathcal{I}g \to A_0\mathcal{I}g \quad B_0\mathcal{I}g \to B_0\mathcal{I}g \quad A_0 = B_0 \to}{A_0\mathcal{I}g, B_0\mathcal{I}g \to g = [A_0B_0]} \qquad \cfrac{C_0\mathcal{I}[A_0B_0] \to C_0\mathcal{I}[A_0B_0]}{C_0\mathcal{I}[A_0B_0] \to}\ (\text{Erase})}{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}g \to}}{A_0\mathcal{I}g, B_0\mathcal{I}g \to (\exists X)(X\not\mathcal{I}g)}$$

□

The cut-elimination procedure[7] yields a cut-free proof of the same end-sequent:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A_0\mathcal{I}g \to B_0\mathcal{I}g \quad B_0\mathcal{I}g \to B_0\mathcal{I}g \quad A_0 = B_0 \to}{A_0\mathcal{I}g, B_0\mathcal{I}g \to g = [A_0B_0]} \qquad \cfrac{C_0\mathcal{I}[A_0B_0] \to C_0\mathcal{I}[A_0B_0]}{C_0\mathcal{I}[A_0B_0] \to}\ (\text{Erase})}{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}g \to}}{\to A_0\not\mathcal{I}g, B_0\not\mathcal{I}g, C_0\not\mathcal{I}g}}{\to (\exists X)(X\not\mathcal{I}g), (\exists X)(X\not\mathcal{I}g), (\exists X)(X\not\mathcal{I}g)}}{\to (\exists X)(X\not\mathcal{I}g)}}{\to (\forall g)(\exists X)(X\not\mathcal{I}g)}$$

♡

[7] or a close look

With the cut-elimination theorem there are some consequences following, e.g. the mid-sequent-theorem for $\mathbf{L_{PG}K}$ and the term-depth-minimization of minimal proofs as found in [5].

5 The Sketch in Projective Geometry

Most of the proofs in projective geometry are illustrated by a sketch. But this method of a graphical representation of the maybe abstract facts is not only used in areas like projective geometry, but also in other fields like algebra, analysis and I have even seen sketches to support understanding in a lecture about large ordinals, which is highly abstract!

The difference between these sketches and the sketches used in projective geometry (and similar fields) is the fact, that proofs in projective geometry deal with geometric objects like Points and Lines, which are indeed objects we can imagine and draw on a piece of paper (which is not necessary true for large ordinals).

So the sketch in projective geometry has a more concrete task than only illustrating the facts, since it exhibits the incidences, which is the only predicate constant besides equality really needed in the formulization of projective geometry. It is a sort of proof by itself and so potentially interesting for a proof-theoretic analysis.

As a first example I want to demonstrate a proof of projective geometry, which is supported by a sketch. It deals with a special sort of mappings, the so called "collineation". This is a bijective mapping from the set of Points to the set of Points, which preserves collinearity. In a formula:

$$\mathrm{coll}(R, S, T) \supset \mathrm{coll}(R\kappa, S\kappa, T\kappa)$$

(Functions in projective geometry are written behind the variables!) The fact we want to proof is

$$\neg\mathrm{coll}(R, S, T) \supset \neg\mathrm{coll}(R\kappa, S\kappa, T\kappa)$$

That means, that not only collinearity but non-collinearity is preserved under a collineation.

The proof is relatively easy and is depicted in fig. 2: If $R\kappa$, $S\kappa$ and $T\kappa$ are collinear, then there exists a Point X' not incident with the Line defined by $R\kappa$, $S\kappa$, $T\kappa$. There exists a Point X, such that $X\kappa = X'$. This Point X doesn't lie on any of the Lines defined by R, S, T. Let $Q = ([RT][XS])$ then $Q\kappa\mathcal{I}[R\kappa S\kappa]$ and $Q\kappa\mathcal{I}[S\kappa X\kappa]$, that is $Q\kappa\mathcal{I}[S\kappa X']$ (since collinearity is preserved). So $Q\kappa = S\kappa$ (since $Q\kappa = ([R\kappa S\kappa][S\kappa X']) = S\kappa$), which is together with $Q \neq S$ a contradiction to the injectivity of κ.

This sketch helps you to understand the relation of the geometric objects and you can follow the single steps of the verbal proof.

If we are interested in the concept of the sketch in mathematics in general and in projective geometry in special then we must set up a formal description

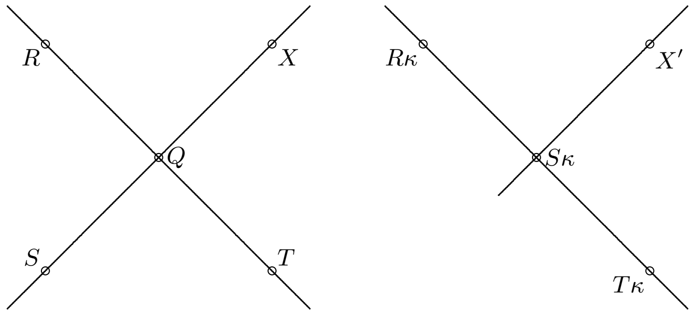

Fig. 2. Sketch of the proof $\neg\text{coll}(R, S, T) \supset \neg\text{coll}(R\kappa, S\kappa, T\kappa)$

of what we mean by a sketch. This is necessary if we want to be more concrete on facts on sketches.

We will give only a short description of what a sketch is and refer the interested reader to [1] for a detailed exposition of the formulization.

A sketch is coded as a quadruple $(\mathcal{M}, \mathcal{E}_+, \mathcal{E}_-, Q)$, where $\mathcal{M}$ is a set of terms with certain limitations, $\mathcal{E}_+$ is a set of positive atomic formulas with the predicate $\mathcal{I}$ over the set $\mathcal{M}$, $\mathcal{E}_-$ is a set of negated atomic formulas with the predicate $\mathcal{I}$ over the set $\mathcal{M}$ and Q is a set of equalities. All these sets have to obey certain requirements ensuring the consistency.

But a sketch is only a static concept, nothing could happen, we cannot "construct". So we want to give some actions on a sketch, which construct a new sketch with more information. These actions on sketches should reflect the actions done by geometrician when drawing, i.e. developing, a sketch. After these actions are defined we can explain what we mean by a construction in this calculus for construction.

The actions primarily operate on the set $\mathcal{E}_+$, since the positive facts are those which are really constructed in a sketch. But on the other hand there are some actions to add negative facts to a sketch. This is necessary for formalizing the elementary way of proving a theorem by an indirect approach.

The actions are:

- Connection of two Points X, Y
- Intersection of two Lines g, h
- Assuming a new Object C in general position
- Giving the Line $[XY]$ a name $g := [XY]$
- Giving the Point (gh) a name $P := (gh)$
- Identifying two Points u and t
- Identifying two Lines l and m
- Using a "Lemma": Adding a positive literal $t\mathcal{I}u$
- Using a "Lemma": Adding a negative literal $t\not\mathcal{I}u$
- Using a "Lemma": Adding a negative literal $t \neq u$

To deduce a fact with sketches we connect the concept of the sketch and the concept of the actions into a new concept called construction. This construction will deduce the facts.

A construction is coded as a rooted and directed tree with a sketch attached to each node[8] and certain demands on its structure.

Finally it is possible to define what a construction deduces by observing the formulas in the leafs of the tree.

We want to mention that great parts of the actions can be automatized so that the constructor can concentrate on the construction. We want to develop a program incorporating these ideas which produces a proof from a sketch.

6 An Example of a Construction

In this section we want to give an example proved on the one hand within $\mathbf{L_{PG}K}$ and on the other hand within the calculus of construction given in the last section. Although a lot of the concepts mentioned in this part weren't introduced, we give the full listing to give the reader a hint on what is really happening.

We want to prove the fact that the diagonal-point $D_1 := ([A_0B_0][C_0D_0])$ and the diagonal-point $D_2 := ([A_0C_0][B_0D_0])$ are distinct. See fig. 3 for the final sketch, i.e. we have already constructed all the necessary objects from the given Points A_0, B_0, C_0, D_0. This step is relatively easy and there are no problems with any of the controls.

The construction tree is depicted in fig. 3 and the respective labels can be found in the table on p. 11. Note the bold formulas in $\mathcal{E}_+^5, \mathcal{E}_-^5$ and in $\mathcal{E}_+^6, \mathcal{E}_-^6$, which yield the contradiction.

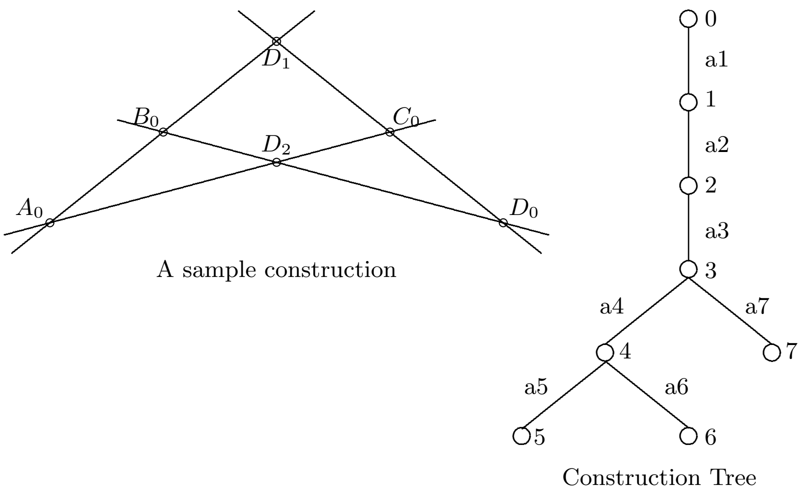

Fig. 3. Sketch and Construction Tree

In the following lists and in the figure not all formulas are mentioned, especially such formulas unnecessary for the construction are not listed. For the

[8] Actually a semisketch, but don't bother about ...

construction tree compare with fig. 3. We can see, that the case-distinction after $D_1 = D_2$ yields a contradiction in any branch, therefore we could deduce with the construction that $D_1 \neq D_2$, since this formula is in all leafs, which are not contradictious.

$$\mathcal{M}^0 = \{A_0, B_0, C_0, D_0, [A_0B_0], \dots\}$$
$$\mathcal{E}^0_+ = \{A_0\mathcal{I}[A_0B_0], \dots\}$$
$$\mathcal{E}^0_- = \{A_0 \neq B_0, \dots, A_0\not\mathcal{I}[C_0D_0]\}$$
$$Q^0 = \{A_0 = A_0, \dots, [A_0B_0] = [A_0B_0], \dots\}$$

$$a1 = \quad ([A_0B_0][C_0D_0])$$
$$\mathcal{M}^1 = \{A_0, B_0, C_0, D_0, [A_0B_0], \dots, ([A_0B_0][C_0D_0])\}$$
$$\mathcal{E}^1_+ = \{A_0\mathcal{I}[A_0B_0], \dots, ([A_0B_0][C_0D_0])\mathcal{I}[A_0B_0], \dots\}$$
$$\mathcal{E}^1_- = \{A_0 \neq B_0, \dots, A_0\not\mathcal{I}[C_0D_0]\}$$
$$Q^1 = Q^0 \cup \{([A_0B_0][C_0D_0])\}$$

$$a2 = \quad ([A_0C_0][B_0D_0])$$
$$\mathcal{M}^2 = \{A_0, B_0, C_0, D_0, [A_0B_0], \dots, ([A_0B_0][C_0D_0]), ([A_0C_0][B_0D_0])\}$$
$$\mathcal{E}^2_+ = \{A_0\mathcal{I}[A_0B_0], \dots, ([A_0B_0][C_0D_0])\mathcal{I}[A_0B_0], ([A_0C_0][B_0D_0])\mathcal{I}[A_0C_0], \dots\}$$
$$\mathcal{E}^2_- = \{A_0 \neq B_0, \dots, A_0\not\mathcal{I}[C_0D_0]\}$$
$$Q^2 = Q^1 \cup \{([A_0C_0][B_0D_0])\}$$

$$a3 = \quad g := [A_0B_0], h := [C_0D_0], l := [A_0C_0], m := [B_0D_0], D_1 := (gh), D_2 := (lm)$$
$$\mathcal{M}^3 = \{A_0, B_0, C_0, D_0, g, h, l, m, D_1, D_2\}$$
$$\mathcal{E}^3_+ = \{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}h, D_0\mathcal{I}h, A_0\mathcal{I}l, C_0\mathcal{I}l, B_0\mathcal{I}m, D_0\mathcal{I}m, D_1\mathcal{I}g, D_1\mathcal{I}h, D_2\mathcal{I}l, D_2\mathcal{I}m\}$$
$$\mathcal{E}^3_- = \{C_0\not\mathcal{I}g, D_0\not\mathcal{I}g, A_0\not\mathcal{I}h, B_0\not\mathcal{I}h, B_0\not\mathcal{I}l, D_0\not\mathcal{I}l, A_0\not\mathcal{I}m, C_0\not\mathcal{I}m, A_0 \neq B_0, \dots\}$$
$$Q^3 = \{A_0 = A_0, \dots, g = g, h = h, l = l, m = m, D_1 = D_1, D_2 = D_2\}$$

$$a4 = \quad D_1 = D_2$$
$$\mathcal{E}^4_+ = \{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}h, D_0\mathcal{I}h, A_0\mathcal{I}l, C_0\mathcal{I}l, B_0\mathcal{I}m, D_0\mathcal{I}m, D_1\mathcal{I}g, D_1\mathcal{I}h, D_1\mathcal{I}l, D_1\mathcal{I}m\}$$
$$\mathcal{E}^4_- = \{C_0\not\mathcal{I}g, D_0\not\mathcal{I}g, A_0\not\mathcal{I}h, B_0\not\mathcal{I}h, B_0\not\mathcal{I}l, D_0\not\mathcal{I}l, A_0\not\mathcal{I}m, C_0\not\mathcal{I}m, A_0 \neq B_0, \dots\}$$
$$Q^4 = \{A_0 = A_0, \dots, g = g, h = h, l = l, m = m, D_1 = D_1, D_2 = D_2, D_1 = D_2\}$$

$$a5 = \quad g = l$$
$$\mathcal{E}^5_+ = \{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}h, D_0\mathcal{I}h, \mathbf{C_0\mathcal{I}g}, B_0\mathcal{I}m, D_0\mathcal{I}m, D_1\mathcal{I}g, D_1\mathcal{I}h, D_1\mathcal{I}m\}$$
$$\mathcal{E}^5_- = \{\mathbf{C_0\not\mathcal{I}g}, D_0\not\mathcal{I}g, A_0\not\mathcal{I}h, B_0\not\mathcal{I}h, B_0\not\mathcal{I}g, D_0\not\mathcal{I}g, A_0\not\mathcal{I}m, C_0\not\mathcal{I}m, A_0 \neq B_0, \dots\}$$
$$Q^5 = \{A_0 = A_0, \dots, g = g, h = h, l = l, m = m, D_1 = D_1, D_2 = D_2, D_1 = D_2, g = l\}$$

$$a6 = \quad A_0 = D_1$$
$$\mathcal{E}^6_+ = \{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}h, D_0\mathcal{I}h, A_0\mathcal{I}l, C_0\mathcal{I}l, B_0\mathcal{I}m, D_0\mathcal{I}m, A_0\mathcal{I}h, \mathbf{A_0\mathcal{I}m}\}$$
$$\mathcal{E}^6_- = \{C_0\not\mathcal{I}g, D_0\not\mathcal{I}g, A_0\not\mathcal{I}h, B_0\not\mathcal{I}h, B_0\not\mathcal{I}l, D_0\not\mathcal{I}l, \mathbf{A_0\not\mathcal{I}m}, C_0\not\mathcal{I}m, A_0 \neq B_0, \dots\}$$
$$Q^6 = \{A_0 = A_0, \dots, g = g, h = h, l = l, m = m, D_1 = D_1, D_2 = D_2, D_1 = D_2, A_0 = D_1, A_0 = D_2\}$$

$$a7 = \quad D_1 \neq D_2$$
$$\mathcal{E}^7_+ = \{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}h, D_0\mathcal{I}h, A_0\mathcal{I}l, C_0\mathcal{I}l, B_0\mathcal{I}m, D_0\mathcal{I}m, D_1\mathcal{I}g, D_1\mathcal{I}h, D_2\mathcal{I}l, D_2\mathcal{I}m\}$$
$$\mathcal{E}^7_- = \{C_0\not\mathcal{I}g, D_0\not\mathcal{I}g, A_0\not\mathcal{I}h, B_0\not\mathcal{I}h, B_0\not\mathcal{I}l, D_0\not\mathcal{I}l, A_0\not\mathcal{I}m, C_0\not\mathcal{I}m, A_0 \neq B_0, \dots, D_1 \neq D_2\}$$
$$Q^7 = \{A_0 = A_0, \dots, g = g, h = h, l = l, m = m, D_1 = D_1, D_2 = D_2\}$$

Table 1. Table of the construction tree nodes

We will now give also a short description of what is happening in this tree: The initial sketch is

$$\begin{aligned}\mathcal{M} &= \{A_0, B_0, C_0, D_0, [A_0B_0], \dots\}\\ \mathcal{E}_+ &= \{A_0\mathcal{I}[A_0B_0], \dots, D_0\mathcal{I}[C_0D_0]\}\\ \mathcal{E}_- &= \{A_0\not{\mathcal{I}}[B_0C_0], \dots, D_0\not{\mathcal{I}}[A_0B_0]\}\end{aligned}$$

After constructing the points D_1 and D_2 and with the shortcuts $[A_0B_0] = g$, $[C_0D_0] = h$, $[A_0C_0] = l$, $[B_0D_0] = m$ we obtain

$$\begin{aligned}\mathcal{M} &= \{A_0, B_0, C_0, D_0, g, h, l, m, D_1, D_2, \dots\}\\ \mathcal{E}_+ &= \{A_0\mathcal{I}g, B_0\mathcal{I}g, C_0\mathcal{I}h, D_0\mathcal{I}h,\\ &\quad A_0\mathcal{I}l, C_0\mathcal{I}l, B_0\mathcal{I}m, D_0\mathcal{I}m,\\ &\quad D_1\mathcal{I}g, D_1\mathcal{I}h, D_2\mathcal{I}l, D_2\mathcal{I}m\}\\ \mathcal{E}_- &= \{C_0\not{\mathcal{I}}g, D_0\not{\mathcal{I}}g, A_0\not{\mathcal{I}}h, B_0\not{\mathcal{I}}h,\\ &\quad B_0\not{\mathcal{I}}l, D_0\not{\mathcal{I}}l, A_0\not{\mathcal{I}}m, C_0\not{\mathcal{I}}m,\\ &\quad A_0 \neq B_0, \dots\}\end{aligned}$$

We now want to add $D_1 \neq D_2$. For this purpose we identify D_1 and D_2 and put the new sets through the contradiction procedure. We will now follow the single steps:

$$D_2 \leftarrow D_1 \tag{1}$$

and as a consequence

$$D_2\mathcal{I}l \Rightarrow D_1\mathcal{I}l \tag{1a}$$

$$D_2\mathcal{I}m \Rightarrow D_1\mathcal{I}m \tag{1b}$$

and so we get the critical constellation $(A_0, D_1; g, l)$

$$A_0\mathcal{I}g, A_0\mathcal{I}l, D_1\mathcal{I}g, D_1\mathcal{I}l \tag{C}$$

Inquiring the first solution $g = l$ yields

$$l \leftarrow g \tag{1.1}$$

and as a consequence

$$C_0\mathcal{I}l \Rightarrow C_0\mathcal{I}g \tag{1.1a}$$

which is a contradiction to

$$C_0\not{\mathcal{I}}g \in \mathcal{E}_- \tag{1.1b}$$

Inquiring the second solution $D_1 = A_0$ yields

$$D_1 \leftarrow A_0 \tag{1.2}$$

and as a consequence

$$D_1 \mathcal{I} m \Rightarrow A_0 \mathcal{I} m \tag{1.2a}$$

which is a contradiction to

$$A_0 \not\mathcal{I} m \in \mathcal{E}_- \tag{1.2b}$$

Since these are all the critical constellations and a contradiction is derived for each branch, the assumption that $D_1 = D_2$ is wrong and $D_1 \neq D_2$ can be added to $\mathcal{E}_-$.

We will now give a proof in $\mathbf{L_{PG}K}$ which corresponds to the above construction. The labels in this proof will not be the rules of $\mathbf{L_{PG}K}$, but references to the above lines.

$\mathit{\Pi}_1$:

$$\dfrac{\dfrac{\overset{(1.1)}{g = l \rightarrow g = l} \quad \rightarrow C_0 \mathcal{I} l}{g = l \rightarrow C_0 \mathcal{I} g}\,(1.1\text{a})}{g = l \rightarrow}\,(1.1\text{b})$$

$\mathit{\Pi}_2$:

$$\dfrac{\dfrac{\overset{(1.2)}{A_0 = D_1 \rightarrow A_0 = D_1} \quad \dfrac{\overset{(1)}{D_1 = D_2 \rightarrow D_1 = D_2} \quad \rightarrow D_2 \mathcal{I} m}{D_1 = D_2 \rightarrow D_1 \mathcal{I} m}\,(1\text{b})}{A_0 = D_1, D_1 = D_2 \rightarrow A_0 \mathcal{I} m}\,(1.2\text{a})}{A_0 = D_1, D_1 = D_2 \rightarrow}\,(1.2\text{b})$$

$\mathit{\Pi}_3$:

$$\dfrac{\dfrac{\dfrac{\rightarrow A_0 \mathcal{I} g \quad \rightarrow A_0 \mathcal{I} l \quad \overset{(1.1)}{g = l \rightarrow g = l}}{\rightarrow g = l, A_0 = (gl)} \quad \dfrac{\rightarrow D_1 \mathcal{I} g \quad \dfrac{\overset{(1)}{D_1 = D_2 \rightarrow D_1 = D_2} \quad \rightarrow D_2 \mathcal{I} l}{D_1 = D_2 \rightarrow D_1 \mathcal{I} l}\,(1\text{a}) \quad \overset{(1.1)}{g = l \rightarrow g = l}}{D_1 = D_2 \rightarrow g = l, D_1 = (gl)}}{D_1 = D_2 \rightarrow g = l, A_0 = D_1}}{D_1 = D_2 \rightarrow g = l \vee A_0 = D_1}$$

$\mathit{\Pi}_1$ examines the branch when $g = l$, $\mathit{\Pi}_2$ examines the branch when $A_0 = D_1$, and $\mathit{\Pi}_3$ deduces that either $g = l$ or $D_1 = A_0$ under the assumption that $D_1 = D_2$ has to be true. The final proof is

$$\dfrac{\overset{\vdots\, \mathit{\Pi}_3}{D_1 = D_2 \rightarrow g = l \vee A_0 = D_1} \quad \dfrac{\overset{\vdots\, \mathit{\Pi}_1}{g = l \rightarrow} \quad \overset{\vdots\, \mathit{\Pi}_2}{A_0 = D_1, D_1 = D_2 \rightarrow}}{g = l \vee A_0 = D_1, D_1 = D_2 \rightarrow}}{D_1 = D_2 \rightarrow}\,(\text{Cut})$$

From this example we can see that construction and proof are very similar in this case. In the next section we want to prove the general result that any construction can be transformed into a proof and vice versa.

7 The Relation between Sketches and Proofs

It is possible to translate a “proof” by construction into a proof in $\mathbf{L_{PG}K}$ and it is also possible to show the equivalence of these to concepts.

Theorem 2. *For any sequent proven in* $\mathbf{L_{PG}K}$ *there is a set of constructions deducing the same sequent and vice versa.*

PROOF: The proof depends on Herbrand’s theorem in the original form and can be found in [1].

According to Tarski (cf. [7]) the abstract theory of projective geometry is undecidable. This result should yield interesting consequences on the relation between sketches and proofs.

8 Closing Comments

We hope that this first analysis of projective geometry from a proof-theoretic point of view opens up a new interesting way to discuss features of projective geometry, which is widely used in a lot of applied techniques. Especially the fact, that the sketches drawn by geometers have actually the same strength as the proofs given in a formal calculus, puts these constructions in a new light. Till now they were considered as nothing more then hints to understand the formal proof by exhibiting you the incidences. But they can be used as proves by themselves.

Therefore we want to develop an automatic sketching tool producing proofs in $\mathbf{L_{PG}K}$ and further analyze $\mathbf{L_{PG}K}$, since e.g. the various interpolation theorems and the discussion of Beth’s definability theorem should yield interesting consequences on projective geometry and the way new concepts are defined in projective geometry.

Furthermore we want to discuss some other systems, especially for a treatise on generalized sketches, which should compare to generalizations of proofs (see [2]).

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