Quantifier Elimination for quantified propositional logics on Kripke frames of type ω

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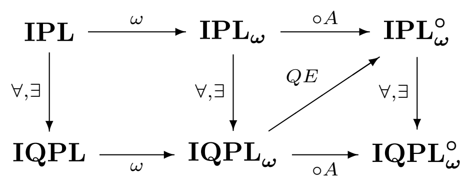
Abstract. The minimal extension of intuitionistic propositional language is characterized, where propositional quantifier are eliminable w.r.t. Kripke frames of type ω.

1 Introduction

In this paper we discuss semantics for quantified propositional logics derived from Kripke frames and demonstrate quantifier elimination for propositional quantifiers w.r.t. Kripke frames of type ω. The language admitting quantifier elimination is the usual language of intuitionistic propositional logic extended by $\circ$, $\circ y$ representing $\forall x(x \vee x \to y)$, the residuum. We derive a sound and complete axiomatization.

1.1 Layout of the article

This paper will present some new logical systems, derived from Intuitionistic Propositional Logic and Intuitionistic Quantified Propositional Logic by restricting the accessibility relation to the order-type ω.



The logic **IPL** Intuitionistic Propositional Logics is well known and well studied. The logic **IQPL** Intuitionistic Quantified Propositional Logic as presented in this article was introduced and studied by Gabbay [Gab81] (as 2h and C2h). We will introduce syntax and semantics for these logics and present a complete axiomatization in Section 2, where also a discussion on different definitions of semantics for Intuitionistic Quantified Propositional Logic is given.

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The logics in the second row are obtained by restricting the possible Kripke frames to ω orderings. Some results about these logics are presented in Section 3.

We extend the language with an additional operator to obtain the logics of the last row and discuss these logics in Section 4. Finally, Section 5 will show that quantifier elimination can be obtained for $\mathbf{IQPL}^{\circ}_{\omega}$ with respect to $\mathbf{IPL}^{\circ}_{\omega}$, thus also obtaining an axiomatization of $\mathbf{IQPL}_{\omega}$.

2 Intuitionistic (Quantified) Propositional Logics (IPL and IQPL)

We will introduce all the definitions for the propositional and the quantified propositional in parallel.

2.1 Syntax and semantics

Definition 1 The basic language L (L^q) consists of the constant $\bot$, countably many propositional variables $\mathcal{X} = (X_1, X_2, \dots)$ and the connectives $\wedge$, $\vee$ and $\to$ (and $\forall$ and $\exists$). The set $\mathrm{Frm}(L)$ ($\mathrm{Frm}(L^q)$) of well formed formulas is defined as the smallest set satisfying the following conditions: $\bot$ and all the X_i are contained in $\mathrm{Frm}(L)$, and if A and B are in $\mathrm{Frm}(L)$ ($\mathrm{Frm}(L^q)$), then also $A \wedge B$, $A \vee B$, $A \to B$ (and $\exists X A$ and $\forall X A$).

Definition 2 We introduce the following notations

$$\begin{aligned} \top &\quad :\leftrightarrow \quad \bot \to \bot \\ \neg A &\quad :\leftrightarrow \quad A \to \bot \\ A \prec B &\quad :\leftrightarrow \quad (B \to A) \to B \\ A \leftrightarrow B &\quad :\leftrightarrow \quad (A \to B) \wedge (B \to A) \end{aligned}$$

The following definitions introduce the semantics for the logics under discussion.

Definition 3 Let (W, R) be a partial order. A subset $X \subseteq W$ is called *upwards closed w.r.t. R* iff for all w, w', if $w \in X$ and $w\ R\ w'$, then $w' \in X$. The set of all subsets of W which are upwards closed w.r.t. R is denoted with $\mathrm{Up}(W, R)$.

Definition 4 A *intuitionistic Kripke frame* is a triple $K = (W, R, \mathcal{P})$, where (W, R) is a partial order, i.e. R is a reflexive, transitive and antisymmetric binary relation on the set W, and $\mathcal{P}$ is a subset of $\mathrm{Up}(W, R)$. As usually we drop the phrase 'intuitionistic' and simply speak about *Kripke frames.* A Kripke frame is *linear* if (W, R) is a linear order.

Definition 5 An *intuitionistic model* is a pair (K, φ) where $K = (W, R, \mathcal{P})$ is a intuitionistic Kripke frame, and φ a mapping from the set of variables $\mathcal{X}$ to $\mathcal{P}$.

The set $\mathcal{P}$ is the set of propositions, i.e. subsets of W where variables and $\bot$ are true in the usual interpretation in Kripke models. It is important to mention that this definition of semantics introduces an additional degree of freedom in choosing the set $\mathcal{P}$, the set of propositions. The usual definition would specify $\mathcal{P}$ as the set of all possible propositions $\mathrm{Up}(W, R)$. This motivates the following definition.

Definition 6 Let $K = (W, R, \mathcal{P})$. A model $M = (K, \varphi)$ with a maximal set of propositions $\mathcal{P} = \mathrm{Up}(K)$ is *complete*, otherwise *partial*.

A model is *safe* if for every formula A there is an X such that $M(A) = \varphi(X)$.

With $M[P/X]$ we will denote the model $M' = (K', \varphi')$ obtained from $M = (K, \varphi)$ where $K' = K$ and $\varphi'(X) = P$, $\varphi'(X') = \varphi(X')$ for $X \neq X'$.

Definition 7 Given a model $M = (K, \varphi)$ and a formula A, we define $M(A)$ as follows:

$$\begin{aligned}
M(X) &= \varphi(X), \text{ for propositional variables } X \\
M(\bot) &= \emptyset \\
M(A \wedge B) &= M(A) \cap M(B) \\
M(A \vee B) &= M(A) \cup M(B) \\
M(A \to B) &= \{w \in W : \forall v \text{ with } w\, R\, v, \text{ if } w \in M(A) \text{ then } v \in M(B)\} \\
M(\forall X A) &= \bigcap \{M[P/X](A) : P \in \mathcal{P}\} \\
M(\exists X A) &= \bigcup \{M[P/X](A) : P \in \mathcal{P}\}
\end{aligned}$$

A formula A is *validated* by M, written $M \Vdash A$, if $W = M(A)$. A formula A is *valid*, if A is validated by all models M.

2.2 Propositional Logic

Definition 8 The set of all valid formulas from $\mathrm{Frm}(L)$ is designated with **IPL**.

Definition 9 We denote with IPL the following deduction system consisting of the axiom schemes

$$\begin{array}{ll}
(P1) & A \to (B \to A) \\
(P2) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\
(P3) & (A \wedge B) \to A; \qquad (A \wedge B) \to B \\
(P4) & A \to (B \to (A \wedge B)) \\
(P5) & A \to (A \vee B); \qquad B \to (A \vee B) \\
(P6) & (A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \\
(P7) & \bot \to A
\end{array}$$

and the rule (MP) from A and $A \to B$ deduce B.

If a formula is derivable using this deduction system we denote this with $\mathsf{IPL} \vdash A$.

In the propositional case the additional distinction in partial and complete models, i.e. the freedom to choose the set of propositions, is superfluous. The proof is straightforward by using a counter example in a partial model as a counter example in a complete model.

Proposition 10 *The propositional logics of complete and partial models coincide.*

The above proposition allows us to state the following completeness theorem without proof, as there are several completeness proofs for intuitionistic propositional logic with respect to Kripke semantics in the literature, e.g. [Tak87].

Theorem 11 *For all formulas* $A \in \mathrm{Frm}(L)$, $\mathbf{IPL} \Vdash A$ *if and only if* $\mathsf{IPL} \vdash A$.

From now on we will use the completeness of **IPL** by asserting that a sentence of **IPL** is provable by demonstrating that it is valid.

2.3 Quantified Propositional Logic

Definition 12 The set of all valid formulas from $\mathrm{Frm}(L^q)$ is designated with $\mathbf{IQPL}^-$. The set of all valid formulas with respect to *safe* models is designated with **IQPL**.

The property of being a safe model induces the validity of the full comprehension scheme and ensures that all images of M actually exist in $\mathcal{P}$, i.e. in a safe mode, $M(A)$ is an element of $\mathcal{P}$ for all A.

The logics $\mathbf{IQPL}^-$ and **IQPL** have been introduced by Gabbay [Gab81], where the following complete axiomatizations were given:

Definition 13 Let IQPL^- denote the deduction system obtained from the deduction system IPL by the extension with the following axiom schemes and rules:

$(Q1)$	$\forall X A(X) \to A(Y)$		(axiom)
$(Q2)$	$A(Y) \to \exists X A(X)$		(axiom)
$(Q3)$	$\dfrac{A(X) \to B}{\exists X A(X) \to B}$,	X not free in B,	(provability rule)
$(Q4)$	$\dfrac{B \to A(X)}{B \to \forall X A(X)}$,	X not free in B,	(provability rule)
$(Q5)$	$\forall X(B \vee A(X)) \to (B \vee \forall X A(X))$		(axiom)

and IQPL denote the deduction system by additionally adding

$(Q6)$	$\exists X(X \leftrightarrow A)$,	X not free in A,	(axiom)

If a formula is derivable using these deduction systems we denote this with $\mathsf{IQPL} \vdash A$ and $\mathsf{IQPL}^- \vdash A$, respectively.

This particular axiom system has been introduced by Gabbay [Gab81] and the full comprehension scheme $(Q6)$ is used to distinguish between safe and unsafe models. Furthermore, the existence of this full comprehension scheme allows us to use the substitution rule:

Proposition 14 *For all formulas $A(X)$ and formulas F from L^q where the free variable of F do not occur in $A(X)$, we can deduce $A(F)$ from $A(X)$ and* IQPL.

Proof. The formula $(X \leftrightarrow F \wedge A(X)) \rightarrow A(F)$ is provable for all A by induction on formulas. From this and $(Q6)$ we obtain the proposition. □

Another option to obtain the substitution rule would have been to add schemata similar to $(Q1)$ and $(Q2)$ with arbitrary formulas for Y. But then we would have to add a distinction on the structure of the formulas Y to draw the distinction between the safe and unsafe models.

In his examination Gabbay proved that the present systems are sound and complete for the respective logics:

Theorem 15 ([Gab81] p. 160, Thm. A & B) *For all $A \in \mathrm{Frm}(L^q)$, $\mathsf{IQPL} \vdash A$ if and only if $\mathbf{IQPL} \Vdash A$, and $\mathsf{IQPL}^- \vdash A$ if and only if $\mathbf{IQPL}^- \Vdash A$.*

This result is in sharp contrast to the following theorem of Kremer:

Theorem 16 ([Kre97]) *The class of all complete models is recursively isomorphic to full second order classical logic.*

The reason for the difference is the additional degree of freedom allowed in choosing sets of propositions for models.

The result of Kremer depends on the presence of wide trees induced by R of complete models $(W, R, \mathcal{P}, \varphi)$, cf. the following theorem of Zach:

Theorem 17 ([Zac04]) *The class of complete models $(W, R, \mathcal{P}, \varphi)$, where R induces trees of arity and width $\leq \omega$, is decidable.*

3 ω-frames

In the following we will restrict the accessibility relation to ω. In these cases we will denote the accessibility relation with $\leq$. Furthermore we will only consider *safe* models, and we will show that the notion of validity with respect to safe and with respect to complete models coincide.

Definition 18 If $(W, \leq)$ has the order-type of ω, then a model based on a the Kripke frame $K = (W, \leq, \mathcal{P})$ is called ω-model. A is *ω-valid* iff A is validated by every *safe* ω-model. The set of all ω-valid sentences from $\mathrm{Frm}(L)$ is designated by $\mathbf{IPL}_{\boldsymbol{\omega}}$. The set of all ω-valid sentences from $\mathrm{Frm}(L^q)$ is designated by $\mathbf{IQPL}_{\boldsymbol{\omega}}$.

Notice that we restrict validity to safe models (comparable to **IQPL**). In the case of ω-models this boils down to the very simple property that W and $\perp$ are in the range of φ:

Lemma 19 *An ω-model $M = (W, R, \mathcal{P}, \varphi)$ is safe iff there are variables $X_\top$ and $X_\bot$ such that $\varphi(X_\top) = W$ and $\varphi(X_\bot) = \emptyset$, and thus also $W, \emptyset \in \mathcal{P}$.*

Proof. By induction on the complexity of formulas: For variables and propositional formulas it is obvious that $M(A) = \varphi(X)$. Consider $\forall X A(X)$:

$$M(A) = \bigcup \{M[P/X](A) : P \in \mathcal{P}\}$$

by induction $M[P/X](A) = P_i \in \mathcal{P}$

$$= \bigcup_{i \in I} P_i$$

which is either one of the P_i or the empty set, thus

$$= \varphi(X) \qquad \text{for an } X \in \mathcal{X}.$$

For $\exists X A(X)$ we get that $M(A)$ is either one of the P_i or W. Thus, the model is safe iff W and $\emptyset$ are in the range of φ. □

Remark 20 In the case of a linear order of type ω we can uniquely specify elements of $\mathrm{Up}(W, \leq)$ by giving the smallest element, which always exists. Thus, we can write every set $P \in \mathrm{Up}(W, \leq)$ as $P = w^\uparrow = \{v : w \leq v\}$.

We extend a deduction system for intuitionistic propositional logic to a deduction system for the logic given above.

Definition 21 We will call the deduction system obtained from IPL extended by the linearity axiom scheme (lin) $(A \to B) \vee (B \to A)$, with IPL_ω.

The following theorem is a consequence of the equivalence of infinite valued propositional Gödel logics and linear Intuitionistic Propositional Logic, as exhibited already in [Baa96], where also the completeness for propositional Gödel logics is shown.

Theorem 22 *For all $A \in \mathrm{Frm}(L)$, $\mathbf{IPL}_{\boldsymbol{\omega}} \Vdash A$ if and only if $\mathsf{IPL}_\omega \vdash A$.*

If we consider propositional logics of general linear orders, not necessarily of type ω, the respective logic $\mathbf{IQPL}_{\mathbf{lin}}$ coincides with $\mathbf{IPL}_{\boldsymbol{\omega}}$.

We will use the completeness of $\mathbf{IPL}_{\boldsymbol{\omega}}$ by asserting the derivability of a sentence by stating its validity.

3.1 Quantified Propositional Logic

Interestingly, it is possible to show that in the ω case the notion of validity with respect to safe and complete models coincide also for quantified propositional logics.

Definition 23 The *filtration* of a partial linear model $(W, \leq, \mathcal{P}, \varphi)$ is a structure $(W', \leq', \mathcal{P}', \varphi')$ where

$$\begin{aligned}
[w] &= \{v : \forall P \in \mathcal{P}\,(v \in P \leftrightarrow w \in P)\} \\
W' &= \{[w] : w \in W\} \\
[w] \leq' [v] &\Leftrightarrow w \leq v \\
\mathcal{P}' &= \{\{[w] : w \in P\} : P \in \mathcal{P}\} \\
\varphi'(X) &= \{[w] : w \in \varphi(X)\}
\end{aligned}$$

Lemma 24 *Let M be an ω-model and M' its filtration. (a) M' is a linear model. (b) If $\mathcal{P}$ is infinite, then M' is an ω-model, otherwise M' is a finite linear model. (c) $M(A) = W$ iff $M'(A) = W'$. (d) If M is safe, then M' is complete.*

Proof. (a) We have to show that (i) $[w]$ is well defined, (ii) $\leq'$ is well defined and a linear ordering, (iii) $\mathcal{P}'$ is a subset of $\mathrm{Up}(W', \leq')$, (iv) $\varphi'(X)$ is well defined.

Ad (i): If $v \in [w]$, then $\forall P \in \mathcal{P}\,(v \in P \leftrightarrow w \in P)$, thus also $w \in [v]$, i.e. $[v] = [w]$. Ad (ii): Well-definedness, reflexivity and linearity are obvious. Antisymmetry: $w \leq v$ iff $(w \in P \to v \in P)$, thus if both $w \leq v$ and $v \leq w$, we have $[w] = [v]$. Ad (iii): We have to show that $\mathcal{P}'$ is upward closed. Assume that $[w] \in P'$, $[w] \leq' [v]$, and that $P' = \{[u] : u \in P\}$. From $[w] \leq' [v]$ we know that $w \leq v$, thus if $w \in P$, then $v \in P$, and together that $[v] \in P'$. Ad (iv): We have to show that $\varphi'(X) \in \mathcal{P}'$, but this is obvious from $\varphi(X) \in \mathcal{P}$ and the definition of $\varphi'(X)$.

(b) Obvious. (c) by straightforward induction.

(d) Assume that $P' \notin \mathcal{P}'$, and that $P' = [w]^{\uparrow}$ (see Remark 20). Furthermore assume that P' is chosen minimal under all elements of $\mathrm{Up}(W', \leq') \setminus \mathcal{P}'$. Thus there is a $Q' \in \mathcal{P}'$ such that Q' is the predecessor of P' (in the worst case this predecessor is W', predecessor and successor in the well defined sense of total linear orders). Let $Q' = [v]^{\uparrow}$. Then $[v] \leq' [w]$. But for all $P \in \mathcal{P}$, it is also true that $v \in P \leftrightarrow w \in P$, thus $[v] = [w]$ and also $[v]^{\uparrow} = Q' = P' = [w]^{\uparrow}$, contradiction. $\square$

Using Lemma 24 we can show that for ω-models it is enough to consider complete models:

Lemma 25 *Every ω-model with infinite $\mathcal{P}$ is isomorph to a complete ω-model.*

Proof. The filtration of an ω-model with infinite $\mathcal{P}$ is again an ω-model, $\mathcal{P}'$ is also infinite, and it is complete. $\square$

We are aiming at the quantifier elimination of propositional quantifiers from $\mathbf{IQPL}_{\boldsymbol{\omega}}$. It will be shown that such an reduction without an extension of the base language cannot be achieved. From now on we will concentrate on ω-models and start with the introduction of the new operator $\circ$. After we have shown quantifier elimination we will derive a complete axiomatization of $\mathbf{IQPL}_{\boldsymbol{\omega}}$ (c.f. Corollary 56).

4 Extensions with $\circ$

4.1 Propositional Logic

Definition 26 The language $L_\circ$ is obtained from L by adding the unary connective $\circ$.

Definition 27 A model for intuitionistic propositional logic with $\circ$ is a ω-model extended with the following definition of M for the operator $\circ$:

$$M(\circ A) = \{n \in \mathbb{N} : n + 1 \in M(A)\}$$

Definition 28 The set of all ω-valid sentences from $\mathrm{Frm}(L_\circ)$ is designated by $\mathbf{IPL}^\circ_{\boldsymbol{\omega}}$.

Definition 29 Let $\mathsf{IPL}^\circ_\omega$ be the Hilbert system obtained from the Hilbert system IPL_ω for $\mathbf{IPL}_{\boldsymbol{\omega}}$ by extending it with the following axioms:
($\circ$a) $\circ\bot \leftrightarrow \bot$
($\circ$b) $\circ(A \star B) \leftrightarrow (\circ A \star \circ B)$ for $\star \in \{\wedge, \vee, \rightarrow\}$.
($\circ$c) $\neg A \vee (A \prec \circ A)$
($\circ$d) $(A \prec B) \rightarrow (\circ A \rightarrow B)$
($\circ$e) $(A \prec \circ B) \rightarrow ((A \rightarrow B) \vee A)$

We will refer to these five axioms ($\circ$a)-($\circ$e) as $\circ$-axioms. We will show that $\mathsf{IPL}^\circ_\omega$ is sound and complete for $\mathbf{IPL}^\circ_{\boldsymbol{\omega}}$, i.e. that $\mathbf{IPL}^\circ_{\boldsymbol{\omega}} \Vdash A$ if and only if A is provable in $\mathsf{IPL}^\circ_\omega$ (in symbols, $\mathsf{IPL}^\circ_\omega \vdash A$).

Theorem 30 (Soundness) *If* $\mathsf{IPL}^\circ_\omega \vdash A$ *then* $\mathbf{IPL}^\circ_{\boldsymbol{\omega}} \Vdash A$.

Proof. By induction on the length of proofs. □

Let A° be the formula which results from A by shifting $\circ$ as far as inside possible. We abbreviate $\circ \ldots \circ A$ (n occurrences of $\circ$) by $\circ^n A$.

Proposition 31 *For every* $n \geq 0$, $\mathsf{IPL}^\circ_\omega \vdash \top \leftrightarrow \circ^n \top$ *and* $\mathsf{IPL}^\circ_\omega \vdash \bot \leftrightarrow \circ^n \bot$.

Proof. By

$$\begin{aligned} \circ\top &\leftrightarrow \circ(\bot \leftrightarrow \bot) && \text{Definition} \\ &\leftrightarrow (\circ\bot \leftrightarrow \circ\bot) && (\circ\text{b}) \\ &\leftrightarrow (\bot \leftrightarrow \bot) && (\circ\text{a}) \\ &\leftrightarrow \top && \text{Definition} \end{aligned}$$

and induction. □

Proposition 32
1. $\mathsf{IPL}^\circ_\omega \vdash A \rightarrow \circ A$
2. $\mathsf{IPL}^\circ_\omega \vdash \neg\neg A \vee (\circ A \rightarrow A)$
3. $\mathsf{IPL}^\circ_\omega \vdash A \leftrightarrow A^\circ$

4. $\mathsf{IPL}^{\circ}_{\omega} \vdash (A \star B) \to (\circ A \star \circ B)$ *for* $\star \in \{\wedge, \vee, \to, \prec\}$
5. $\mathsf{IPL}^{\circ}_{\omega} \vdash \neg\neg A \leftrightarrow ((\circ^{k+n+1} A \to \circ^k A) \to \bigwedge_{v=0}^{n} \circ^{k+v} A)$
6. $\mathsf{IPL}^{\circ}_{\omega} \vdash \neg\circ^m A \to \neg\circ^n A$ *(for all* $n \leq m$*)*
7. $\mathsf{IPL}^{\circ}_{\omega} \vdash F \leftrightarrow ((\neg A \wedge F) \vee (\neg\neg A \wedge F))$

Proof. 1. The following equations are valid in **IPL**, and due to the completeness also provable in IPL:

$$\begin{aligned} \top &\leftrightarrow A \to \top && \text{by } \mathsf{IPL} \\ &\leftrightarrow A \to (\bot \to \circ A) && \text{by } \mathsf{IPL} \\ &\leftrightarrow (A \to \bot) \to (A \to \circ A) && \text{by } \mathsf{IPL} \\ &\leftrightarrow \neg A \to (A \to \circ A) && \text{by definition of } \neg \end{aligned}$$

From ($\circ$c) $\neg A \vee (A \prec \circ A)$, and by IPL $\neg A \vee (A \to \circ A)$. By IPL, $(A \to \circ A) \vee (A \to \circ A)$, so $(A \to \circ A)$.

2. From (P7) $\bot \to A$ and the definition of $\neg A$ as $A \to \bot$ we obtain that $\neg A \to (A \leftrightarrow \bot)$, using ($\circ$b) and Proposition 31 ($\circ\bot \leftrightarrow \bot$) we obtain (*) $\neg A \to (\circ A \leftrightarrow \bot)$. Again from the definition of the negation we have $\neg A \to (\bot \to A)$ where we can substitute for $\bot$ the formula $\circ A$ due to (*), thus $\neg A \to (\circ A \to A)$. Using the theorem of $\mathbf{IPL}_{\boldsymbol{\omega}}$ that $\neg A \vee \neg\neg A$ and by IPL, $\neg\neg A \vee (\circ A \to A)$.
3. by induction on the complexity of formulas, using ($\circ$b).
4. from (1) and ($\circ$b)
5. Let $F = (\circ^{k+n+1} A \to \circ^k A) \to \bigwedge_{v=0}^{n} \circ^{k+v} A$. We will show that $\neg A \to (F \leftrightarrow \neg\neg A)$ and $\neg\neg A \to (F \leftrightarrow \neg\neg A)$ from which we obtain $(F \leftrightarrow \neg\neg A)$. First for $\neg A \to (F \leftrightarrow \neg\neg A)$: As in the proof of 2. we see that $\neg A \to (\circ^n A \leftrightarrow \bot)$ for all n, and in a similar way, that $\neg A \to (\neg\neg A \leftrightarrow \bot)$, which is equivalent to $\neg A \to (\neg\neg A \leftrightarrow (\bot \to \bot) \to \bigwedge_{v=0}^{n} \bot)$ where we then fill in the right $\circ^n A$ to obtain $\neg A \to (\neg\neg A \leftrightarrow F)$.
Concerning $\neg\neg A \to (F \leftrightarrow \neg\neg A)$: The one direction is an axiom, the other direction $\neg\neg A \to (\neg\neg A \to F)$ is equivalent to $\neg\neg A \to F$. Expanding $\prec$ in axiom ($\circ$c) together with the assumption $\neg\neg A$ we obtain $\neg\neg A \to ((\circ A \to A) \to \circ A)$, thus also $\neg\neg A \to ((\circ A \to A) \to ((\circ A \to A) \wedge \circ A))$ and $\neg\neg A \to ((\circ A \to A) \to A)$. Using this, induction on n and the shifting of $\circ$ over all propositional connectives we obtain the result.
6. From axiom ($\circ$c) we obtain $\neg A \vee (\neg\circ A \to \neg A)$, together with the axiom $\neg A \to (\neg\circ A \to \neg A)$ we deduce $\neg\circ A \to \neg A$. Together with the shifting of $\circ$ we obtain the result.
7. The following equivalences are trivial consequences of the axioms:

$$F \leftrightarrow (\top \wedge F) \leftrightarrow ((\neg A \vee \neg\neg A) \wedge F) \leftrightarrow ((\neg A \wedge F) \vee (\neg\neg A \wedge F)) \qquad \square$$

Definition 33 Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a finite set of variables and let

$$\Gamma_m^{\mathcal{X}} = \{\circ^k X : X \in \mathcal{X}, 0 \leq k \leq m\} \cup \{\bot, \top\}.$$

A $\circ$-*chain on* X is an expression

$$(\bot \bowtie_1 A_1) \wedge (A_1 \bowtie_2 A_2) \wedge \ldots \wedge (A_{n(m+1)} \bowtie_{n(m+1)} \top)$$

where $\bowtie_i \in \{\leftrightarrow, \prec\}$, $\bowtie_i = \prec$ for at least one i, all the A_i are different, and

$$\{\bot, A_1, \ldots, A_{n(m+1)}, \top\} = \Gamma_m^{\mathcal{X}}.$$

The set of all chains based on $\mathcal{X}$ is denoted with $\mathcal{C}(\mathcal{X})$.

The notion $\mathcal{X}(A)$ for any formula A denotes the set of variables occurring in A.

We say that A and B from $\Gamma_m^{\mathcal{X}}$ are equivalent w.r.t. to a $\circ$-chain C of there are $B_1, \ldots, B_n$ such that $A \leftrightarrow B_1$, $B_1 \leftrightarrow B_2, \ldots, B_n \leftrightarrow B$ (or their symmetric equivalents) are conjuncts in C. The set of all B equivalent to A w.r.t. C is denoted by $[A]_C$. Obviously, the equivalence of formulas in $\Gamma_m^{\mathcal{X}}$ w.r.t. C is an equivalence relation, and the $[A]_C$ provide a partition of $\Gamma_m^{\mathcal{X}}$.

The equivalence relation induced by $[A]_C$ also provides an ordering of the equivalence classes in the natural way: $[A]_C \bowtie [B]_C$ iff $A \bowtie B$. Exhibiting the well-definedness is straight-forward.

Definition 34 A $\circ$-chain on $\mathcal{X}$ is said to be an *m-chain on $\mathcal{X}$* if the additional conditions hold:

1. $\circ^i X$ is in the equivalence class of $\bot$ iff all $\circ^k X$ for $0 \leq k \leq m$ are in the equivalence class of $\bot$.
2. Assume there are l equivalence classes and that $X \in \mathcal{X}$ is in the k-th equivalence class. Then $\circ^i X$ is in the $k+i$-th equivalence class if $k + i \leq l$, and in the equivalence class of $\top$ otherwise.

The set of all m-chains based on $\mathcal{X}$ is denoted with $\mathcal{C}_m(\mathcal{X})$.

Remark 35 Henceforth we will freely identify chains which coincide w.r.t. equivalence classes. Furthermore, we will use m-chains as definition of models (resp. counter examples) in the following way: Choose $\varphi(X) = \varphi(Y)$ for all $X \in [Y]_C$. Furthermore, if $[X]_C \prec [Y]_C$ then choose $\varphi(X) \subset \varphi(Y)$. It is easy to show by induction that this defines a model.

m-chains extend chains used in [Baa96] to provide a finitary completeness proof of **IPL** plus $(A \to B) \vee (B \to A)$ with respect to linearly ordered Kripke semantics.

Lemma 36 (Order Theoretic 'Tertium non datur') *For all $\mathcal{X}$ and m,*

$$\mathsf{IPL}_\omega^\circ \vdash \bigvee_{C \in \mathcal{C}_m(\mathcal{X})} C$$

where $\mathcal{C}_m(\mathcal{X})$ is the set of all m-chains over $\mathcal{X}$.

Proof. The formula $(A \prec B) \vee (A \leftrightarrow B) \vee (B \prec A)$ is valid in $\mathbf{IPL}_{\boldsymbol{\omega}}$, thus also provable from IPL_ω. Substituting formulas containing $\circ$ does not destroy this property of being provable. Thus by Proposition 45

$$\mathsf{IQPL}_\omega \vdash \bigwedge_{A_i, A_j \in \Gamma_m^{\mathcal{X}}} (A_i \prec A_j \vee A_i \leftrightarrow A_j \vee A_j \prec A_i).$$

Distribute this formula to a disjunction of conjunctions. Replace $\bot \to A$ with $\top$, $A \to \top$ with $\top$. Using $(\top \to \bot) \Leftrightarrow \bot$ and transitivity of $\prec$ and $\to$, we obtain a disjunction of $\circ$-chains, i.e. an ordering of the equivalence classes of $\Gamma_m^{\mathcal{X}}$, where $\bot$ is the minimal class, $\top$ is the maximal class and there are at least two classes. We have to transfer those $\circ$-chains into m-chains, i.e. reducing the number of 'violations' of properties (1) and (2) from Definition 34.

We use the correction steps (a)-(c) below to ensure (1) and (2). Every correction step reduces the number of violations or transforms the $\circ$-chain into $\bot$.

(a) If $\circ^k X$ occurs in the equivalence class of $\bot$ and $\circ^l X$ occurs in a higher equivalence class, then we delete the $\circ$-chain C (from $\neg\circ^k X \to \neg\circ^l X$ (Proposition 32.6), and $C \to \neg\circ^k X$ we get $C \to \neg\circ^l X$, but also $C \to \neg\neg\circ^l A$).
All remaining $\circ$-chains fulfill (1).

(b) If $\circ^{k+l+1} X$ occurs in an equivalence class r not containing $\bot$, with r less or equal to the equivalence class of $\circ^k X$, then we adjoin all equivalence classes $\geq r$ to the equivalence class of $\top$ ($C \to \circ^{k+l+1} X \to \circ^k X$, $C \to \neg\neg\circ^k X$, $C \to \circ^k X \prec \circ^{k+l+1} X$ and using the scheme $(A \prec B) \wedge (B \to A) \to (A \wedge B)$).

(c) If $\circ^k X$ occurs in the equivalence class r, $\circ^{k+1} X$ occurs in the equivalence class of $r + 2 + i$ for some $i \geq 0$, then we adjoin all equivalence classes $\geq r$ to the equivalence class of $\top$ ($C \to \circ^k X \prec E \wedge E \prec \circ^{k+1} X$ it follows $C \to \circ^{k+1} X \to E \wedge E \prec \circ^{k+1} X$ by axiom (e)).

(d) If $\top$ and $\bot$ are in the same class, we delete the $\circ$-chain C. □

We are now ready to prove the completeness of $\mathsf{IPL}^\circ_\omega$ with respect to $\mathbf{IPL}^\circ_\omega$:

Theorem 37 (Completeness) *If* $\mathbf{IPL}^\circ_\omega \Vdash A$ *then* $\mathsf{IPL}^\circ_\omega \vdash A$.

Proof. Let A° consist of variables $\mathcal{X} = \{X_1, \dots, X_n\}$ and components of $\circ^k X$, $X \in \mathcal{X}$ with $k \leq m$. Let C be an m-chain, i.e. $C \in \mathcal{C}_m(\mathcal{X})$. By induction on formulas we can show that for all $E, F \in \Gamma_m^{\mathcal{X}}$:

$$\begin{array}{lcl}
\mathsf{IPL}^\circ_\omega \vdash C \to (\neg E \leftrightarrow \top) & \text{or} & \mathsf{IPL}^\circ_\omega \vdash C \to (\neg E \leftrightarrow \bot) \\
\mathsf{IPL}^\circ_\omega \vdash C \to (E \to F \leftrightarrow \top) & \text{or} & \mathsf{IPL}^\circ_\omega \vdash C \to (E \to F \leftrightarrow F) \\
\mathsf{IPL}^\circ_\omega \vdash C \to (E \wedge F \leftrightarrow E) & \text{or} & \mathsf{IPL}^\circ_\omega \vdash C \to (E \wedge F \leftrightarrow F) \\
\mathsf{IPL}^\circ_\omega \vdash C \to (E \vee F \leftrightarrow E) & \text{or} & \mathsf{IPL}^\circ_\omega \vdash C \to (E \vee F \leftrightarrow F)
\end{array}$$

Therefore, $C \to (A \leftrightarrow D)$ for some $D \in \Gamma_m^{\mathcal{X}}$ by Proposition 32.3. D is called the *evaluation* of A under C ($\mathrm{val}_C(A)$). The valuation of a formula is well-defined, and for valid formulas A, $\mathrm{val}_C(A)$ is contained in the equivalence class of $\top$, because otherwise C provides a counterexample (see Remark 35).

Now let A be a tautology, we know from Lemma 36 that

$$\mathsf{IPL}^\circ_\omega \vdash \bigvee_{C \in \mathcal{C}} C$$

where $\mathcal{C}$ is the set of all m-chains over X. From this we obtain

$$\begin{array}{lll}
 & \mathsf{IPL}^\circ_\omega \vdash \bigvee_{C\in\mathcal{C}} (C \wedge \top) & \text{by } \mathsf{IPL} \\
\Leftrightarrow & \mathsf{IPL}^\circ_\omega \vdash \bigvee_{C\in\mathcal{C}} (C \wedge \mathrm{val}_C(A)) & \text{since } \mathrm{val}_C(A) \in [\top]_C \\
\Leftrightarrow & \mathsf{IPL}^\circ_\omega \vdash \bigvee_{C\in\mathcal{C}} (C \wedge A) & \text{by } \mathsf{IPL} \\
\Leftrightarrow & \mathsf{IPL}^\circ_\omega \vdash (\bigvee_{C\in\mathcal{C}} C) \wedge A & \text{by } \mathsf{IPL} \\
\Leftrightarrow & \mathsf{IPL}^\circ_\omega \vdash A & \text{by } \mathsf{IPL}
\end{array}$$

□

Corollary 38 *For every A either*

$$\mathsf{IPL}^\circ_\omega \vdash A \leftrightarrow \bot$$

or there is a subset $\varDelta \subseteq \mathcal{C}(\mathcal{X}(A))$ such that

$$\mathsf{IPL}^\circ_\omega \vdash A \leftrightarrow \bigvee_{C\in\varDelta} C.$$

Proof. The formula $C \wedge \mathrm{val}_C(A)$ where $\mathrm{val}_C(A) \notin [\top]_C$ collapses to another chain C' by adjoining all equivalence classes greater or equal to the equivalence class of $\mathrm{val}_C(A)$. □

Remark 39 The completeness is a *weak* one, i.e. it shows that all tautologies are derivable. But the system is not strongly complete, i.e. with respect to entailment, as the entailment is not compact [BZ98].

4.2 Quantified Propositional Logic

Definition 40 The language $L^q_\circ$ is obtained from L^q by adding the unary connective $\circ$.

Definition 41 The set of all ω-valid sentences from $\mathrm{Frm}(L^q_\circ)$ is designated by $\mathbf{IQPL}^\circ_\omega$.

Definition 42 Let $\mathsf{IQPL}^\circ_\omega$ be the Hilbert system obtained from the Hilbert system IQPL_ω for $\mathbf{IQPL}_\omega$ by extending it with the $\circ$-axioms (see Def. 29, p. 8) and the axioms

$$\begin{array}{ll}
(\mathrm{nx}) & \exists X(\circ X \leftrightarrow A) \\
(\mathrm{dis}) & \neg\forall X(X \vee \neg X)
\end{array}$$

Thus, the system $\mathsf{IQPL}^\circ_\omega$ consists of the rules for intuitionistic propositional logic (Def. 9), together with linearity, the Q-axioms (Def. 13), the $\circ$-axioms (Def. 29) and the two axioms from above.

Remark 43 The axiom (dis) is not valid if we do consider models which are not complete. But due to Lemma 24 (c) we can restrict ourselves to complete models. Furthermore, note that the logic $\mathbf{IQPL}_{\boldsymbol{\omega}}$ is not an intermediate logic.

By induction using (nx) we obtain the following Lemma:

Lemma 44 $\mathsf{IQPL}_\omega^\circ \vdash \exists X(\circ^k X \leftrightarrow A)$ *for all* k.

Due to the fact that we consider only complete models, it is easy to show by induction on proofs that the substitution property holds also for the extended language:

Proposition 45 *If* $\mathsf{IQPL}_\omega^\circ \vdash A(X)$ *and the free variables of* F *do not occur bound in* $A(X)$*, then also* $\mathsf{IQPL}_\omega^\circ \vdash A(F)$.

We will prove soundness and completeness of this system via quantifier elimination in Section 5. This is not a trivial corollary and it seems that it will be hard to get a different completeness proof.

Theorem 46 *For all formulas* $A \in \mathrm{Frm}(L_\circ^q)$, $\mathbf{IQPL}_{\boldsymbol{\omega}}^\circ \Vdash A$ *if and only if* $\mathsf{IQPL}_\omega^\circ \vdash A$.

The proof of this theorem will be given at the end of the next section.

5 Quantifier Elimination for $\mathbf{IQPL}_{\boldsymbol{\omega}}^\circ$

We reduce a formula from $\mathrm{Frm}(L_\circ^q)$ to $\mathrm{Frm}(L_\circ)$.

Remark 47 We will use the following method several times in proofs: If we have to exhibit that

$$\mathsf{IQPL}_\omega^\circ \vdash (E \wedge \exists X A(X)) \leftrightarrow (E \wedge B)$$
$$(\mathsf{IQPL}_\omega^\circ \vdash (E \wedge \forall X A(X)) \leftrightarrow (E \wedge B))$$

we proceed as follows: For the left-to-right (right-to-left) direction we search for a suitable tautology $(E \wedge A(X)) \to (E \wedge B)$ (resp. $(E \wedge B) \to (E \wedge A(X))$), which can be derived due to the propositional completeness as shown above. Using the quantifier introduction rule we obtain $E \wedge \exists X A(X)$ (resp. $E \wedge \forall X A(X)$).

For the right-to-left (left-to-right) direction we introduce the existential quantifier (resp. instantiate the universal quantifier).

Proposition 48 *(α) Let* $H(X)$ *be*

$$\neg\neg A \wedge \neg\neg B \wedge (A \prec X) \wedge (X \prec \circ X) \wedge \ldots \wedge (\circ^{k-1} X \prec \circ^k X) \wedge$$
$$\wedge (\circ^k X \leftrightarrow B) \wedge \ldots \wedge (\circ^{k+l} X \leftrightarrow \circ^l B)$$

and $H'(X)$ *be*

$$\neg\neg A \wedge \neg\neg B \wedge (\circ^k A \prec B) \wedge (\circ^k X \leftrightarrow B),$$

then

$$\mathsf{IQPL}_\omega^\circ \vdash H(X) \leftrightarrow H'(X).$$

(β) Let $G(X)$ be

$$\neg A \wedge \neg\neg B \wedge (A \prec X) \wedge (X \prec \circ X) \wedge \ldots \wedge (\circ^{k-1} X \prec \circ^k X) \wedge$$
$$\wedge (\circ^k X \leftrightarrow B) \wedge \ldots \wedge (\circ^{k+l} X \leftrightarrow \circ^l B)$$

and $G'(X)$ be

$$\neg A \wedge \neg\neg B \wedge (\circ^k X \leftrightarrow B),$$

then

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash G(X) \leftrightarrow G'(X).$$

Proof. (α) $\rightarrow$: $\mathsf{IQPL}^{\circ}_{\omega} \vdash A \prec X \rightarrow \circ^k A \prec \circ^k X$, deletion. $\leftarrow$: $\mathsf{IQPL}^{\circ}_{\omega} \vdash (F \leftrightarrow G) \rightarrow (\circ F \leftrightarrow \circ G)$, thus, the equivalence classes of $X, \circ X, \ldots, \circ^{k-1} X$ have to be between A and $\circ^k X$.

(β) $\rightarrow$: deletion. $\leftarrow$: $\mathsf{IQPL}^{\circ}_{\omega} \vdash F \leftrightarrow G \rightarrow \circ F \leftrightarrow \circ G$, as B is non-zero (i.e., $\neg\neg B$ is provable: $\mathsf{IQPL}^{\circ}_{\omega} \vdash \neg\neg B$), $X, \circ X, \ldots, \circ^k X$ have to be non-zero, too. □

Remark 49 Note that if we write $H(x)$ as $\neg\neg A \wedge F$, we can write $G(x)$ as $\neg A \wedge F$, thus,

$$H(x) \vee G(x) \leftrightarrow (\neg\neg A \wedge F) \vee (\neg A \wedge F) \leftrightarrow F$$

according to Proposition 32.7.

Lemma 50 *For all quantifier-free A there is a quantifier-free A' whose free variables are among the free variables of $\exists X A$, such that*

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash \exists X A \leftrightarrow A'.$$

Proof. We start with constructing for A an equivalent disjunction of chains $\bigvee_{C \in \Delta} C$ (or $\bot$) (see Corollary 38) and distribute the existential quantifier over the $\bigvee$. Thus, it suffices to consider formulas of the form $\exists X C$, where C is an m-chain, or $\exists x \bot$, which can be replaced by $\bot$.

Case (1): Assume that X, $\circ X$, $\ldots$, $\circ^m X$ occur in the equivalence class of $\bot$. Then $\mathsf{IQPL}^{\circ}_{\omega} \vdash \exists X C(X) \leftrightarrow C(\bot)$, which can be trivially deduced from $\bot \leftrightarrow X$, which is contained in C, and the substitution rule.

Case (2): If not (1), assume the X, $\circ X$, $\ldots$, $\circ^m X$ occur all in *singular* equivalence classes, i.e. classes which do not contain any other formula. Thus, the chain looks like

$$C \leftrightarrow C' \wedge (E \prec X) \wedge (X \prec \circ X) \wedge \ldots \wedge (\circ^m X \prec G) \wedge C''$$

for some E and G. We know from Proposition 32.7 that

$$\exists X C \leftrightarrow (\neg E \wedge \exists X C) \vee (\neg\neg E \wedge \exists X C)$$

and will prove

$$\neg E \wedge \exists X C \leftrightarrow \neg E \wedge C' \wedge E \prec G \wedge C''$$
$$\neg\neg E \wedge \exists X C \leftrightarrow \neg\neg E \wedge C' \wedge \circ^{m+1} E \prec G \wedge C''$$

which allows the elimination of the quantifier. Consider the former formula: Since $\neg E \wedge C \leftrightarrow \neg E \wedge C' \wedge E \prec G \wedge C''$ is a tautology, it is provable, and thus, also the former formula. The same procedure succeeds for the later formula.

Case (3): Neither (1) nor (2) holds. Let k be the smallest number such that the equivalence class of $\circ^k X$ is not singular, and let B be a formula from this equivalence class. From Proposition 48 and Remark 49 we know that

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash \exists X C \leftrightarrow \exists X H(X) \quad \text{or} \quad \mathsf{IQPL}^{\circ}_{\omega} \vdash \exists X C \leftrightarrow \exists X G(X)$$

Using the axiom $\mathsf{IQPL}^{\circ}_{\omega} \vdash \exists X(B \leftrightarrow \circ^k X)$ (from the axiom $\exists X(\circ X \leftrightarrow A)$) we prove

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash \exists X H(X) \leftrightarrow H'^{\star} \quad \text{and} \quad \mathsf{IQPL}^{\circ}_{\omega} \vdash \exists X G(X) \leftrightarrow G'^{\star}$$

where $H'^{\star}$ ($G'^{\star}$) arise from H' (G') by deleting $B \leftrightarrow \circ^k X$ (Lemma 44). □

Corollary 51 *Assume*

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X A(X, \circ^k Z) \leftrightarrow A'(\circ^k Z)$$

for some free variable Z and some k, A and A' quantifier free, and assume that all free variables in $A'(\circ^k Z)$ occur in $\forall X A(X, \circ^k Z)$. Then for all G there is a quantifier free A_G, where all free variables in A_G occur in $\forall X A(X, \circ^k Z)$ and

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X A(X, G) \leftrightarrow A_G$$

Proof. By Lemma 44 and Lemma 50:

$$\begin{aligned}
&\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X A(X, G) \leftrightarrow \exists Z(G \leftrightarrow \circ^k Z) \wedge \forall X A(X, G) \\
&\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X A(X, G) \leftrightarrow \exists Z((G \leftrightarrow \circ^k Z) \wedge \forall X A(X, \circ^k Z)) \\
&\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X A(X, G) \leftrightarrow \exists Z((G \leftrightarrow \circ^k Z) \wedge A'(\circ^k Z)) \\
&\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X A(X, G) \leftrightarrow A_G
\end{aligned}$$

□

Lemma 52 *Fix a finite set of variables $\mathcal{X}$. Every $\circ$-chain $C \in \mathcal{C}(\mathcal{X})$ is equivalent to a formula where every variable $X \in \mathcal{X}$ only occur in at most 2 atomic formulas, and these formulas are of the form $\circ^k X \lhd E$ or $E' \lhd \circ^k X$, where $\lhd \in \{\prec, \to\}$, and the formulas E and E' are of the form $\circ^l Z$ with $Z \neq X$.*

Proof. Case (1): Assume X occurs in the equivalence class of $\bot$, then $\mathsf{IQPL}^{\circ}_{\omega} \vdash C \leftrightarrow (X \to \bot) \wedge F_C$, where F_C does not contain X.

Case (2): If not (1), assume that $X, \ldots, \circ^m X$ occur all in *singular* equivalence classes, thus, the chain looks like

$$C \leftrightarrow C' \wedge E \prec X \wedge X \prec \circ X \wedge \ldots \wedge \circ^m X \prec G \wedge C''$$

then

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash C \leftrightarrow (E \prec X) \wedge \circ^m X \prec G \wedge F_C$$

where F_C does not contain X, since this is tautologically equivalent.

Case (3): Neither (1) nor (2) holds. Let $\circ^k X$ be the formula with minimal k such that some B is in its equivalence class. Thus, the chains looks like

$$C' \wedge (A \prec X) \wedge (X \prec \circ X) \wedge \ldots \wedge (\circ^{k-1} X \prec \circ^k X) \wedge \\ \wedge (\circ^k X \leftrightarrow B) \wedge \ldots \wedge (\circ^{k+l} X \leftrightarrow \circ^l B) \wedge C''$$

which is tautologically equivalent to

$$(\neg A \wedge \neg\neg B \wedge C) \vee (\neg\neg A \wedge \neg\neg B \wedge C)$$

due to Proposition 32.7 and the fact that X is in the same equivalence class as B, but not in the equivalence class of $\bot$ (Case (1)), thus, also B is not in the equivalence class of $\bot$. Now we can reduce the two disjunction terms of the above formula according to Proposition 48 which yields

$$\neg A \wedge \neg\neg B \wedge (\circ^k X \leftrightarrow B)$$

for the former and

$$\neg\neg A \wedge \neg\neg B \wedge (\circ^k A \prec B) \wedge (\circ^k X \leftrightarrow B)$$

for the latter. These two formulas can be combined using Proposition 32.1 to

$$\neg\neg B \wedge (\circ^k A \prec B) \wedge (\circ^k X \leftrightarrow B)$$

which can be written as

$$F_C \wedge (\circ^k X \rightarrow B) \wedge (B \rightarrow \circ^k X).$$

□

Lemma 53 *For all quantifier-free A there is a quantifier-free A' whose free variables are among the free variables of $\forall X A$, such that*

$$\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X A \leftrightarrow A'.$$

Proof. We start with constructing for A an equivalent disjunction of chains as in Lemma 50. Then we use Lemma 52 to obtain a disjunction of expressions where X occurs only in the form $\circ^k X \rightarrow D$, $D \rightarrow \circ^k X$, $\circ^k X \prec D$, $D \prec \circ^k X$. We treat these expressions and F_C of Lemma 52 as atomic formulas, i.e. we do not expand the definitions, but treat these formulas as if $\prec$ would be part of the language. Distributing $\forall$ over $\wedge$ and confining the range of the quantifier, we arrive at formulas

$$\forall X((E_1 \lhd_1 \circ^{k_1} X) \vee \ldots \vee (E_n \lhd_n \circ^{k_n} X) \vee \\ (\circ^{l_1} X \lhd'_1 G_1) \vee \ldots \vee (\circ^{l_v} X \lhd'_v G_v) \vee \\ (\circ^{m_1} X \lhd''_1 \circ^{n_1} X) \vee \ldots \vee (\circ^{m_w} X \lhd''_w \circ^{n_w} X))$$

where $\lhd_i, \lhd'_j, \lhd''_m \in \{\rightarrow, \prec\}$. The relations where X occurs on both sides ($\circ^{m_i} X \lhd''_i \circ^{n_i} X$) can be reduced to either $\top$ or $\top \rightarrow \circ^s X$ (for some s) depending on the relation of m_i and n_i. Further confining the range of the quantifier we obtain formulas of the form

$$\forall X((E_1 \lhd_1 \circ^{k_1} X) \vee \ldots \vee (E_n \lhd_n \circ^{k_n} X) \vee \\ (\circ^{l_1} X \lhd'_1 G_1) \vee \ldots \vee (\circ^{l_v} X \lhd'_v G_v))$$

Using $(E_i \to E_j) \vee (E_j \to E_i)$ respectively $(G_i \to G_j) \vee (G_j \to G_i)$ all inequalities are reducible to at most one upper and one lower inequality in the following way: Assume for the sake of explanation that we have

$$\forall X(E \to \circ^2 X) \vee (F \to \circ^3 X) \vee (X \to G).$$

Using the Proposition 32.4 we get

$$\forall X(\circ E \to \circ^3 X) \vee (F \to \circ^3 X) \vee (X \to G),$$

introducing the case distinction of linearity

$$[(\circ E \to F) \vee (F \to \circ E)] \wedge \forall X(\circ E \to \circ^3 X) \vee (F \to \circ^3 X) \vee (X \to G).$$

Distributing the conjunction and reducing the implications we arrive at

$$(\circ E \to F) \wedge \forall X(E \to \circ^2 X \vee X \to G) \\ \vee \\ [(F \to \circ E) \wedge \forall X(F \to \circ^3 X \vee X \to G)]$$

In a similar way we can reduce all pairs of inequalities, i.e. occurrences of formulas $F \lhd G$, of the same kind to only one occurrence. Thus, we arrive at the following eight cases:

1. $\forall X(E \prec \circ^l X)$
2. $\forall X(E \to \circ^l X)$
3. $\forall X(\circ^k X \prec G)$
4. $\forall X(\circ^k X \to G)$
5. $\forall X((E \to \circ^l X) \vee (\circ^k X \to G))$
6. $\forall X((E \prec \circ^l X) \vee (\circ^k X \to G))$
7. $\forall X((E \to \circ^l X) \vee (\circ^k X \prec G))$
8. $\forall X((E \prec \circ^l X) \vee (\circ^k X \prec G))$

which we will prove in turn: According to Lemma 44 we may assume, that G is of the form $\circ^k G'$.

(ad 1) $\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X(E \prec \circ^l X) \leftrightarrow \bot$,

(ad 2) $\mathsf{IQPL}^{\circ}_{\omega} \vdash \forall X(E \to \circ^l X) \leftrightarrow \neg E$, both trivially proven by substituting $\bot$ for X.

(ad 3) Using Proposition 32.7 we split up the formula (7):

$$\forall X(\circ^k X \prec G)$$
$$\updownarrow$$
$$\forall X(\circ^k X \prec \circ^k G')$$
$$\updownarrow$$
$$(\neg\circ^k G' \wedge \forall X(\circ^k X \prec \circ^k G')) \vee (\neg\neg\circ^k G' \wedge \forall X(\circ^k X \prec \circ^k G'))$$

What is left is to show that

$$\neg\circ^k G' \wedge \forall X(\circ^k X \prec \circ^k G') \quad \leftrightarrow \quad \bot$$
$$\neg\neg\circ^k G' \wedge \forall X(\circ^k X \prec \circ^k G') \quad \leftrightarrow \quad \neg\neg\circ^k G' \wedge \circ^k G' \quad (\leftrightarrow \circ^k G')$$

are provable. First consider the former formula: For the left-to-right direction substituting $\top$ for X yields $\neg\circ^k G' \wedge (\top \prec \circ^k G')$, which is tautologically equivalent to $\bot$. The right-to-left direction is trivial, since $\bot$ implies everything.

Now consider the latter formula: For the left-to-right direction substituting G' for X yields $\neg\neg\circ^k G' \wedge (\circ^k G' \prec \circ^k G')$, which implies the given right side. For the right-to-left direction consider, that the right side implies $\neg\neg\circ^k G' \wedge ((\circ^k G' \to \circ^k X) \to \circ^k G')$, from which the left hand side follows by quantifier introduction.

(ad 4) Again using Proposition 32.7 we can assume

$$\forall X(\circ^k X \to G)$$
$$\updownarrow$$
$$\forall X(\circ^k X \to \circ^k G')$$
$$\updownarrow$$
$$(\neg\circ^k G' \wedge \forall X(\circ^k X \to \circ^k G')) \vee (\neg\neg\circ^k G' \wedge \forall X(\circ^k X \to \circ^k G'))$$

Using the same argumentation as above, but substituting in the second case not G' but $\circ G'$ instead we can deduce the following equivalences using the scheme $\neg u \wedge (v \prec u) \leftrightarrow \bot$:

$$\neg\circ^k G' \wedge \forall X(\circ^k X \to \circ^k G') \quad \leftrightarrow \quad \bot$$
$$\neg\neg\circ^k G' \wedge \forall X(\circ^k X \to \circ^k G') \quad \leftrightarrow \quad \neg\neg\circ^k G' \wedge \circ^k G'$$

and obtain, again as above,

$$\forall X(\circ^k X \to \circ^k G') \leftrightarrow \neg\neg\circ^k G' \wedge \circ^k G'$$

(α) Dealing with (5)–(8), we again distinguish according to $\neg\circ^k G'$ and $\neg\neg\circ^k G'$, and in every case

$$\neg\circ^k G' \wedge \forall X((E \lhd \circ^l X) \vee (\circ^k X \prec \circ^k G')) \tag{1}$$

reduces to (by the same scheme as above)

$$\neg\circ^k G' \wedge \forall X(E \lhd \circ^l X)$$

by substituting $\top$ as above, yielding a variant of (1) or (2).

(β) Now consider

$$\neg\circ^k G' \wedge \forall X(E \to \circ^l X \vee \circ^k X \to \circ^k G') \tag{2}$$

and let $v = \max\{l, k\}$. The following are provable equivalences

$$\begin{aligned}
(2) \quad &\leftrightarrow \quad \neg\circ^k G' \wedge \forall X(E \to \circ^v X \vee \neg\circ^v X) \\
&\leftrightarrow \quad \neg\circ^k G' \wedge (E \to \forall X(\circ^v(X \vee \neg X))) \\
&\leftrightarrow \quad \neg\circ^k G' \wedge (E \to \circ^v \forall X(X \vee \neg X)) \qquad (4) \\
&\leftrightarrow \quad \neg\circ^k G' \wedge (E \to \circ^v \bot) \\
&\leftrightarrow \quad \neg\circ^k G' \wedge \neg E
\end{aligned}$$

The first implication is obtained from the tautological equivalence of the two formulas without quantifier.

(γ) Considering $\neg\circ^k G' \wedge \forall X(E \prec \circ^l X \vee \circ^k X \to \circ^k G')$ we again make a case distinction after $\neg\circ^k G'$ and define the following two formulas

$$\begin{aligned}
\Delta \quad &:\leftrightarrow \quad \neg E \wedge \neg\circ^k G' \wedge \forall X(E \prec \circ^l X \vee \circ^k X \to \circ^k G') \\
\Gamma \quad &:\leftrightarrow \quad \neg\neg E \wedge \neg\circ^k G' \wedge \forall X(E \prec \circ^l X \vee \circ^k X \to \circ^k G')
\end{aligned}$$

Therefore,

$$\neg\circ^k G' \wedge \forall X(E \prec \circ^l X \vee \circ^k X \to \circ^k G') \leftrightarrow \Delta \vee \Gamma$$

We want to reduce Δ and Γ, first consider Δ:

$$\begin{aligned}
\neg\neg X \vee \neg X, \quad &(\neg\neg X \vee \neg X) \to (\neg\neg\circ^l X \vee \neg\circ^k X), \\
&\neg E \wedge (\neg\neg\circ^l X \vee \neg\circ^k X) \to \neg E \wedge (E \prec \circ^l X \vee \circ^k X \to \circ^k G')
\end{aligned}$$

thus,

$$\Delta \leftrightarrow \neg E \wedge \neg\circ^k G'.$$

Considering Γ, we can repeat the computation from paragraph (β) above for

$$\forall X(E \prec \circ^l X \vee \circ^k X \to \circ^k G') \to (1)$$

which, together with eq. 4, yields $\neg E$, thus

$$\Gamma \leftrightarrow \neg\circ^k G' \wedge \neg\neg E \wedge \neg E \leftrightarrow \bot.$$

Now we are ready to finish the proofs for the remaining cases (5) to (8).

(ad 5 and 6) We have already dealt with

$$(\neg\circ^k G' \wedge \forall X(E \lhd \circ^l X \vee \circ^k X \to \circ^k G'))$$

in paragraphs β an γ for $\lhd =\to$ and $\lhd =\prec$, respectively.

Considering on the other hand

$$(\neg\neg\circ^k G' \wedge \forall X(E \lhd \circ^l X \vee \circ^k X \to \circ^k G'))$$

we can prove

$$\neg\neg\circ^k G' \wedge \forall X(E \lhd \circ^l X \vee \circ^k X \to \circ^k G')$$
$$\updownarrow$$
$$\neg\neg\circ^k G' \wedge (E \lhd \circ^{l+1} G' \vee \circ^k G')$$

by using the scheme $(E \lhd \circ H \to E \lhd F \vee F \to H)$ and substituting $\circ G'$ for X for the $\downarrow$ direction, and for the $\uparrow$ direction, by considering that $\Delta_{(\circ G')} \to \Delta_X$ is a tautology.

(ad 7 and 8) We have already dealt with

$$\neg\circ^k G' \wedge \forall X(E \lhd \circ^l X \vee \circ^k X \prec \circ^k G')$$

in paragraph α. Now consider

$$\forall X(\neg\neg\circ^k G' \wedge (E \lhd \circ^l X \vee \circ^k X \prec \circ^k G'))$$

It is easy to see that with setting

$$\Gamma(X) \leftrightarrow \neg\neg\circ^k G' \wedge (E \lhd \circ^l X \vee \circ^k X \prec \circ^k G')$$

we can prove

$$\forall X(\neg\neg\circ^k G' \wedge (E \lhd \circ^l X \vee \circ^k X \prec \circ^k G')) \leftrightarrow \forall X \Gamma(X)$$
$$\updownarrow$$
$$\neg\neg\circ^k G' \wedge (E \lhd \circ^l G' \vee \circ^k G') \leftrightarrow \Gamma(G')$$

by substituting G' for X for the $\downarrow$ direction, and for the $\uparrow$ direction, by considering that $\Gamma(G') \to \Gamma(X)$ is a tautology.

This completes the proof of the lemma. □

As a consequence of Lemma 50 and Lemma 53 we state the central theorem:

Theorem 54 *Quantified propositional logic of linearly ordered well founded Kripke frames of type ω, $\mathbf{IQPL}_{\boldsymbol{\omega}}$, admits quantifier elimination.*

Corollary 55 *(1)* $\mathsf{IQPL}^{\circ}_{\omega} \vdash A$ *or* $\mathsf{IQPL}^{\circ}_{\omega} \vdash \neg A$ *for all closed formulas.*
(2) $\mathsf{IQPL}^{\circ}_{\omega} \vdash A \vee \neg A$ *for all closed formulas.*

Proof. (ad 1) $\mathsf{IQPL}^{\circ}_{\omega} \vdash A \leftrightarrow A'$, A' quantifier free and variable free, thus, $\mathsf{IQPL}^{\circ}_{\omega} \vdash A'' \leftrightarrow A'$ where A'' consists only of $\top$, $\bot$, $\wedge$, $\vee$, $\to$.

(ad 2) consequence of (1). □

Corollary 56 *The* $\circ$*-free fragment of* $\mathbf{IQPL}_\omega^\circ$*, that is* $\mathbf{IQPL}_\omega$*, is axiomatized by* $\mathsf{IQPL}_\omega^\circ$ *when all occurrences of* $\circ$ *are replaced by its definition:*

$$\circ A \Leftrightarrow \forall X(X \vee X \to A). \tag{5}$$

Proof. If $\mathsf{IQPL}_\omega \vdash A$, then also $\mathsf{IQPL}_\omega^\circ \vdash A$. Within the proof in this system we can substitute all occurrences of $\circ A$ as given above in eq. 5. The substitutions of all the axioms and rules can be proven, thus we obtaining a proof in IQPL_ω.□

This provides also a proof of Theorem 46.

6 Conclusion

Studies in quantified propositional logics can be considered as approaches to refine the semantic-syntax relation. Quantified propositional formulas can be used to discriminate between various adequate concepts of semantics. Consider e.g. $\exists x \forall y((y \to x) \to (\neg y \vee x \leftrightarrow y))$[1] expressing in Kripke notation that there is at least one branch with a maximal world. This sentence is valid in all finite Kripke structures, but not valid in the general case. It is therefore unwise to specify one single quantified propositional logic as quantified propositional intuitionistic logic.

References

[Baa96] M. Baaz. Infinite-valued Gödel logics with 0-1-projections and relativizations. In P. Hájek, editor, *Proc. Gödel'96, Logic Foundations of Mathematics, Computer Science and Physics – Kurt Gödel's Legacy*, LNL 6, pages 23–33. Springer, 1996.

[BZ98] M. Baaz and R. Zach. Compact propositional Gödel logics. In *28th International Symposium on Multiple-valued Logic. May 1998, Fukuoka, Japan. Proceedings*, pages 108–113. IEEE Press, Los Alamitos, 1998.

[Gab81] Dov M. Gabbay. *Semantical Investigations in Heyting's Intuitionistic Logic*, volume 148 of *Synthese Library*. D. Reidel Publishing Company, 1981.

[Kre97] Philip Kremer. On the complexity of propositional quantification in intuitionistic logic. *J. of Symbolic Logic*, 62(2):529–544, June 1997.

[Tak87] Gaisi Takeuti. *Proof Theory.* North Holland, 1987.

[Zac04] R. Zach. Decidability of quantified propositional intuitionistic logic and S4 on trees of height and arity $\leq \omega$. *Jornal of Philosophical Logic*, 2004. to appear.

[1] Of course no purely universal (existential) sentence exists because purely universal (existential) sentences express propositional validity (satisfiability.)